谨以此书
献给我国著名科学家
钱学森同志



钱 学 森



1991年国务院、中央军委授予钱学森"杰出贡献科学家"荣誉称号



江泽民主席在授予钱学森荣誉称号仪式上致辞，高度评价了钱学森对我国科技事业作出的贡献（1991年）



钱学森和夫人蒋英在授予荣誉称号的仪式上（1991年）



授予荣誉称号仪式后，江泽民主席向钱学森同志热烈祝贺（1991 年）



1938 年，钱学森在美国从事应用力学研究。

1939 年，钱学森在美国加州理工学院获航空、数学博士学位

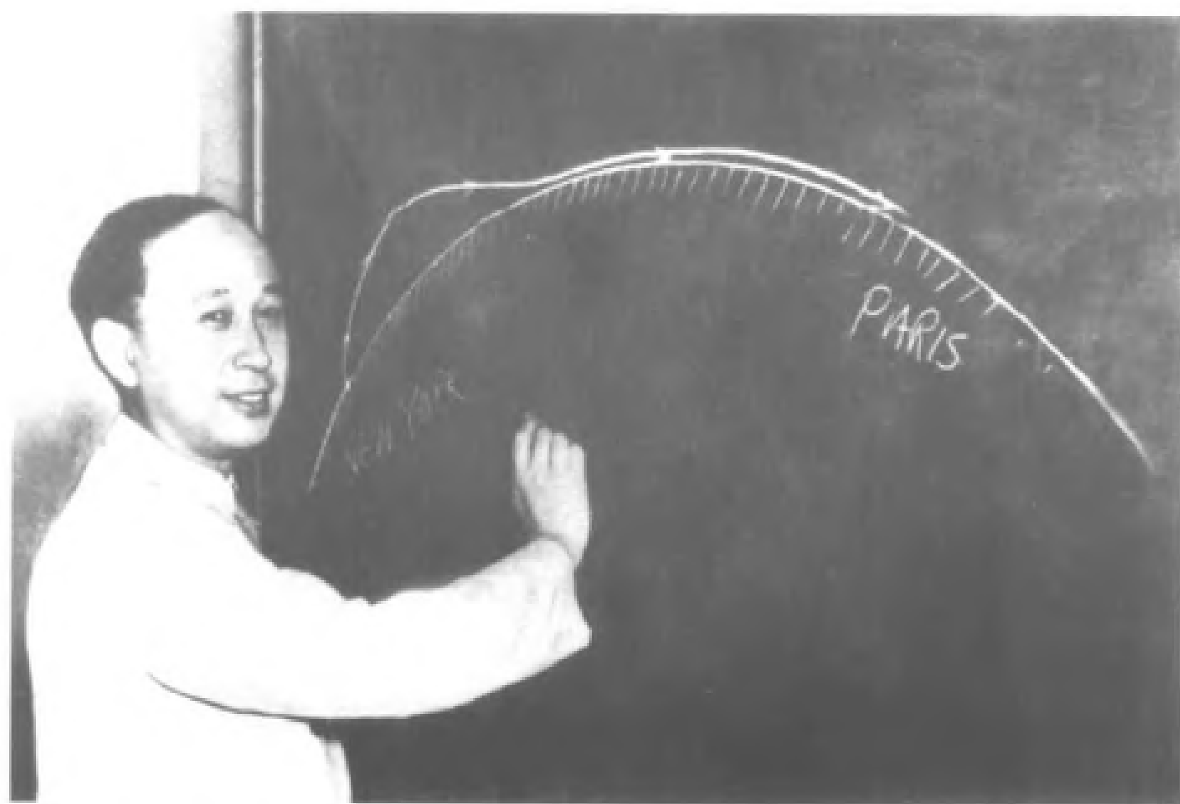




1943 年，钱学森和几位中国研究生在帕萨迪那（Pasadena）与周培源夫妇合影



1949 年，钱学森在加州理工学院授课



钱学森在加州理工学院给研究生讲课：关于远程商用喷气飞机作洲际飞行的问题(时间大约是 1941 年或 1942 年)。



1949 年 10 月 27 日，重返加州理工学院担任喷气推进中心主任的钱学森与同事在办公室留影，右一为 F·马勃(F. Marble)。



1949 年，钱学森在加州理工学院任教，左一为年轻时的 F·马勃(F. Marble)。



1944 年 12 月，担任火箭研究理论组组长的钱学森在美军试验基地(Leech Spring)参加美军最初的火箭"列兵A"(Private A)发射的试验工作 右二为钱学森



1949 年 4—5 月间，第二次世界大战结束前夕，钱学森随冯·卡门率领的科学考察团赴德国考察航空与火箭研究的发展情况，这是钱学森(中)和他的老师冯·卡门(右)在德国哥廷根会见空气动力学家 L·普朗特(L. Prandtl) 普朗特曾是冯·卡门的老师，这是师生三代在战后会见的一个有意义的时刻



钱学森的老师 T. von Kármán 教授



钱学森和几位亚裔同事与冯·卡门及其妹妹在冯·卡门家中合影。前左下蹲者是钱学森，前右下蹲者为谈家桢(时间约为1938年)。



1955年，钱学森一家乘克莱夫兰总统号(President Cleveland)轮船回国。



1950年，钱学森在加州理工学院古根海姆航空实验室。右二为F. Marble。



1989 年国际技术与技术交流大会授予钱学森"小罗克韦尔奖章"和"世界级科学与工程名人"、"国际理工研究所名誉成员"称号。这是钱学森在国内接受奖章和荣誉证书时留影。



耄耋之年的钱学森仍在孜孜不倦地探索科学技术的前沿发展



钱学森夫妇和F. Marble 夫妇合影(1991 年)



钱学森在看F. Marble赠送给他的近期论文，右一为郑哲敏院士，右二为庄逢甘院士(1991年)



钱学森夫人蒋英和F. Marble夫人Ora Lee见面，格外高兴(1991年)



1996 年，钱学森会见从美国来访的老友F. Marble 教授，并交谈学术问题。



1996 年，钱学森和F. Marble 教授亲切交谈



1996年，国防科工委科技委秘书长王寿云研究员(左)和科学院力学研究所薛明伦研究员(右)在钱学森手稿交接仪式上和F. Marble夫妇合影



钱学森手稿交接仪式后，王寿云和薛明伦向F. Marble教授深表谢意(1996年)



1996 年 12 月 11 日、江泽民主席看望 85 岁高龄的钱学森、并亲切交谈。



1999年9月18日中共中央、国务院、中央军委授予钱学森“两弹一星功勋奖章”。这是授奖大会后，全国政协副主席、著名核物理学家朱光亚（左二），中央军委委员、总装备部部长曹刚川上将（右一），总装备部政委李继耐上将（左一）在钱学森家，将“两弹一星功勋奖章”及获奖证书呈交钱老（右二是钱老的夫人、著名声乐家蒋英教授）



本书展出的手稿（包括部分打字稿）选自钱学森1938－1955年在美国从事教学和科研活动时的大量原始资料。

由钱学森在美国加州理工学院同事和挚友Frank E. Marble（弗朗克·马勃）教授精心收集和长期负责保管的手稿共有15000余页，涉及的内容十分广泛，其中：有已发表或未发表论文的手稿、图表、公式推导、演算稿、数据列表等；有多种内部报告的手稿；有对多个科学问题的分析与计算；有风洞设计的手稿、图表等；有与博士论文导师、后来的合作者与领导人Theodore von Kármán（Th. 冯·卡门）及与其他科学家的通信；有听课和自学的笔记；有就某些专题所收集的资料汇总及分析；有给他所指导的研究生的便笺等等。限于篇幅，本书只选择了其中极少一部分。钱学森那个时期公开发表的论文已由科学出版社于1991年编辑成册出版发行。手稿不同于这些论文。论文是科研成果的集中表现，而手稿则反映作者创造性探索的动态过程。前者的读者对象主要是相关专业的科技专家，其中部分内容因其对科学技术发展的重要作用而被永久地载入科学技术的文库；后者因其能生动地表现一位杰出科学家的治学精神和治学态度而为更广泛的读者所关注，特别是对中青年科学家和青少年有极好的教育作用。另外，作为我国一位杰出科学家的工作记录和中华民族优秀文化遗产的一部分，这些材料还有很高的收藏价值。这就是出版这本手稿选编的主要目的。

一

因为手稿全部是用英文写的，为了帮助读者更好地理解《选编》的内容，谨就我们所知，在这里提供一些有关的历史背景情况。

本世纪上半叶是科学和工程走向密切结合的时代，从而形成了技术科学这样一个中间层次的学科，其中最具代表性的一个范例，就是以技术科学为指导把航空工业建立在科学理论的可靠基础上，使飞机设计在空气动力学理论的指导下，突破了“声障”和“热障”，实现了高速飞行，同时发展了火箭技术，开创了航天事业的新时代。当时美国的加州理工学院古根海姆（Guggenheim）航空实验室（GALCIT）就处于推动这项进步的前沿，而它的领导人正是钱学森的老师、后来成为密切合作者的著名学者Theodore von

Kármán（Th·冯·卡门）。年轻的钱学森1934年从上海交通大学机械系铁道机械工程专业毕业，有志于从事航空工程。在考取清华大学公费留美后，1935年来到美国麻省理工学院，次年转学到加州理工学院航空系，从师于von Kármán。钱学森在1939结束博士论文后，除去一个短时期外，他的教学和科研工作基本上都是在加州理工学院进行的。

天资聪慧、受过严格家庭和学校教育、勤奋好学、勇于开拓的钱学森到加州理工学院不久，便显示出令人瞩目的才华。从他在美国20年的论著看，他始终如一地以推动航空和航天新技术的发展为目标，努力探索处于科学与技术最前沿的问题。早在1937年，在从事博士论文研究的同时，他就参加了由同学Frank Malina（弗朗克·马林纳）组织、得到von Kármán支持的火箭技术研究小组，从此，开始了他与火箭和航天技术的不解之缘。如果说他的早期研究主要是针对阻碍当时航空、航天技术发展的一些关键力学问题，那么后来，他的视野更加广泛，前瞻性更强，着眼点已不限于个别问题，而是开辟新的学科前沿领域，以推动航空、航天技术整体与长远的发展了。与此同时，他的学科领域也已不限于应用力学，而是他所倡导的更为广阔得多的技术科学领域了。他在“超级空气动力学”（Superaerodynamics）、《工程控制论》和“物理力学”方面的论文和专著就是清楚的说明。

从事这样的工作并取得杰出的成就，除了个人的天赋之外，还必须经受严格的科学训练，付出极其辛勤的劳动。只有这样才能取得广泛而深刻的知识，也才能在反复的认识、再认识的艰苦过程中，克服一个个困难，最后取得满意的结果和如此杰出的成就。同时，还需要有敢于向未知或知之甚少的领域开拓的精神，需要有敢于向传统观念挑战的勇气

在本世纪30年代，火箭在技术上或理论上都是很不成熟的，并且常常由于被用来和科幻小说中的登月和宇宙航行相联系，而被蒙上了一层神秘的外衣。直到那时，火箭的研究，除极个别情况外，还远没有被纳入传统科学研究的议程。因此选择它作为严肃科学研究的对象，是冒着很大风险的，没有一种向未知领域挑战的严肃科学精神是做不到的。取得如此杰出的成就，更需要持之以恒的动力，而这个动力来自钱学森献身于以科学推动工程与技术的使命感和为中华民族争光争气的民族责任感和民族自豪感。

认真品味和欣赏钱学森的手稿，人们不仅会对他那清秀整洁的卷面赞叹不已，而且会深深体会到他的这种敬业精神和高尚情操。1978年在悼念因公牺牲的著名科学家郭永怀烈士的纪念会上，钱学森所讲的以下一段话，既是对亡友的深切怀念，也体现了他在科学研究中崇高的思想境界，这段话是：

“一方面是精深的理论，一方面是火热的斗争，是冷与热的结合，是理论与实践的结合。这里没有胆小鬼的藏身处，也没有自私者的活动地；这里需要的是真才实学和献身精神。”

二

1947年钱学森回国省亲，在当时的浙江大学、上海交通大学和清华大学做了以工程科学（即现在我们所称呼的技术科学）和超级空气动力学为题的学术报告。我们从他的手稿中分别选取了这两个方面的一些材料。这里特别要讲一讲他对技术科学的观点，因为发展技术科学是他历来的主张，而且也是他的一个重要的学术思想和科学上的追求。

在人类文明的历史上，曾经有很长一段时间，自然科学（包含数学）与工程技术是沿着各自的道路分别发展的。虽然两者之间客观上有着深刻的联系，而且有的人既是科学家也是工程师，但是科学与技术被认为是两种不同的行业。那时自然科学追求的是客观世界的规律，而工程技术主要靠实践得到的经验来满足社会对物质生产的需要。

社会经济的发展和国力的增强需要依靠工业与工程技术的发展，而自然科学的理论可以加速它们的发展，这种自觉的认识始于20世纪初，钱学森在他的报告里对此做了详细的介绍。由于当时的工业与自然科学中的力学关系最为密切，于是自然科学与工程的结合最早便以应用力学的形式在德国出现，并很快在世界上的发达国家得到推广。

历史证明，这种认识是十分正确的。除航空工业外，还有一些其他突出的例子。以稍后原子弹的研究历史为例，爱因斯坦确立了质量与能量间转换的原理和定量规律，物理学家和化学家发现了裂变物质，并且从原理上指出制造原子弹的可能性。然而原子弹的实

现，却是自然科学家与工程师为了一个共同的工程目标，遵循技术科学发展的规律，经过不断探索和密切合作才取得的结果，这进一步体现了技术科学的重要作用。二次世界大战期间，雷达技术也是沿着这样的思路发展的。

工程技术发展的这类实例逐渐使更多的人认识到，在自然科学和工程技术之间，存在着一个相对独立的，以自然科学成果为指导、以解决工程技术问题为目的的科学领域，这就是技术科学。

钱学森自 1936 年起，先是在应用力学的德国学派一代大师 von Kármán 门下学习，后来又长期在这个环境中以应用力学为手段，致力于推动航空工业的发展和开创火箭技术，并且有机会深入了解到美国和其他一些国家航空工业和火箭技术当时与未来发展的蓝图。1936－1945 年，他还目睹了原子弹和雷达的发展。因此，他对技术科学有系统而深刻的了解和独特的见解

他认为在自然科学与工程技术之间，客观上所存在着的技术科学将二者联系起来。为了促进工程技术的发展和增强国力，应当着力发展这一个科学领域。为此，他对技术科学的目标和性质、技术科学特有的观点和方法论，它在当时所包括或所应包括的内容以及技术科学教育等，做了全面而深刻的阐述。应当说这是对技术科学最好的概括，起到了对技术科学界定的历史作用

下面几段话引自钱学森发表于 1948 年 "Engineering and Engineering Sciences" 一文的前言，精辟地概括了技术科学的性质和根本目标：

"人们回顾半个世纪来人类社会的进步，无不对技术和科学研究的重要性，作为国家和国际事务的一个决定性因素，所受重视程度的巨大提高有深刻的印象。很显然，虽然在早期，技术与科学研究是以未加计划的、个体的方式进行的，可是到了今天，在任何主要国家这种研究都是受到认真调控的。因而，如同长期以来的农业、金融政策或者外交关系一样，技术与科学研究现已成为国家的事情。认真考察研究工作的重要性得到如此重视的原因，自然地会得出这样的答案，即研究工作现在是现代工业整体中的一个组成部分，不提到研究工作就谈不上现代工业。既然工业是国家富强的基础，技术和科学研究就是国家富强的关键"。

“人们也许会说，在工业时代的开创时期，技术和科学研究就与工业发展有关，那么为什么今天把研究工作说得如此重要？这个问题的答案是，出于国内和国际竞争的需要，现代工业必须以越来越高的速度发展。做到如此高的发展速度，就必须大大强化研究工作，把出基础科学的发现几乎马上用上去。也许，没有什么比把战时雷达和核能的发展作为例子更为突出的了。雷达技术和核能的成功开发为盟方取得第二次世界大战的胜利做出了重要贡献是公认的事实。短短数年，紧张的研究工作把基础物理学的发现，通过实用的工程，变成了战争武器的成功应用。这样，纯科学上的事实与工业应用间的距离现在很短了 换句话说，长头发纯科学家和短头发工程师的差别其实很小，为了使工业得到有成效的发展，他们间的密切合作是不可少的”。

“纯科学家与从事实用工作的工程师间密切合作的需要，产生了一个新的行业—工程研究家或工程科学家。他们成为纯粹科学和工程之间的桥梁。他们是将基础科学知识应用于工程问题的那些人……”

从以上文字，读者可以看到，50 年前，钱学森就提出技术和科学研究“是国家富强的关键”，作者的爱国情操和洞察能力跃于纸上。在这里，钱学森认为，技术科学首先是服务于工程技术的。它为工程技术提供新原理、新概念、新目标、新途径、新方法、新技术等系统的理论基础与基础技术，促进和带动新产业和高技术的建立和发展。为了达到这样的目的，它必须充分掌握自然科学的最新成果，并深刻了解工程中存在的基本问题。因为工程师们面临的是多因素、复杂的实际问题，技术科学家必须善于从这些问题中找到主要矛盾，创立有充分自然科学依据的、能被工程师用于设计的、有预测能力的定量理论。所以技术科学家的目标是建立近似的实用理论，当发现自然科学的已有成果不够用时，也需要吸收和运用工程中经验性的规律和判断。所以技术科学在这一点上不同于自然科学。另一方面，技术科学又不同于工程技术，因为它的中心目的是研究和解决某类工程技术中带有普遍性的问题，而主要不是一个个具体的工程技术问题。钱学森还认为，数学和计算数学作为一种工具占有十分主要的地位。

钱学森不仅提倡技术科学，而且是身体力行的。1947 年他回

国做学术报告所选的两个题目，表面上看联系不大，实际上超级空气动力学(Superaerodynamics)正是技术科学中的一个很好的典型，因为它把分子运动论和空气动力学联系起来，论述了用于处理高空飞行问题的理论框架。这种情况贯穿于钱学森1947年以后的许多工作中。例如，作为最早认识到自动控制技术在火箭技术中重要作用的一位专家，他既创造性地研究了火箭发动机和火箭飞行控制的多种问题，又写出了为工程控制论这一学科奠基的专著。又如，为了深入研究火箭发动机，他应用统计力学、光谱学和化学动力学，研究了气体和液体的平衡和输运性质，又以技术科学的观点为美国加州理工学院的研究生开了名为物理力学的课程，这时其内容已不限于气体的力学问题了，而是开辟了技术科学的一个新的分支。

总之，大力发展技术科学是钱学森的一个基本主张。理解了这一点，就能较好地解释，为什么他的研究领域有这样大的变化，为什么他努力从具体问题的研究提高到新学科的建立。他的目的是要为带动工程技术的发展，提供超前性的技术科学理论基础。本世纪后半叶科学技术的发展表明，他所提出的主张和倡导的技术科学分支具有很强的预见性。

三

为了体现钱学森学术思想的发展和学术领域的开拓，编者从他所从事的各个领域里，选择了一些代表性手稿的片断，虽说这样做肯定不可能反映那个时期他所从事工作的全貌，但我们仍然相信，读者能从中体验到他献身科学技术的执著精神、严密的治学态度、创造性的成就和所涉及的广泛领域。

他的研究领域包括以下四个方面，即应用力学、喷气推进、工程控制论和物理力学。应用力学又包含空气动力学和固体力学两个大的分支学科。书中选用的材料取自他15000余页的手稿。由于本书的目的并非反映专业内容，因此只就若干专题摘取了其中的少数几页。在每份材料之前，编者加了一些简短说明，目的是帮助非专业的读者了解所涉及的科学问题，以及这个问题在推动航空和喷气技术发展中所处的地位。

读者可以看到，一方面他的研究工作所涉及的科学问题是十分宽广的，另一方面又紧紧地瞄准了航空和火箭技术发展的需

要。同时他并不满足于仅仅解决一个个具体的科学问题，哪怕这些问题有多么重要。他总是在研究了这些问题之后，随即提出前瞻性极强、带有方向性的问题，并对之进行深入而系统的研究，为未来工程技术的发展指出新的方向。他的关于核动力推动的火箭的论文和关于用火箭推进的远程商用运输机的论文，便是这方面很典型的例子。当他发现一些自然科学的基础知识对解决一类技术问题不够用时，便根据工程技术的需要和特点，系统地从其他自然科学汲取必要的知识，使其成为工程科学的一个新的分支。他的论文"Superaerodynamics – Rarefied Gas Dynamics"（超级空气动力学——稀薄气体力学）以及他的专著《工程控制论》和《物理力学讲义》，是他长期研究空气动力学，火箭发动机和火箭飞行轨道控制和优化，发动机的热力学、热化学和热空气动力学后的研究成果，这些著作刻画了新的技术科学领域。1946年由他编著而以美国 Air Technical Service Command（空军技术后勤司令部）名义出版的、做为内部教材的《Jet Propulsion》（喷气推进）一书，是美国第一部全面和系统地论述火箭与喷气推进科学技术的专著，内容有基本理论直到包括从导弹射程、制导和通信在内的喷气推进技术应用的众多方面。它是加州理工学院在喷气推进技术方面多年研究工作的总结与提高。本选编中的许多理论研究的内容在这本书中作了系统的反映。

所选手稿突出地表现出他的清秀、工整的字体，按照严格标准书写的运算方程和计算公式，以及规范化的列图制表等特征。这些特征贯穿于他的全部手稿，不论它们是来自草稿、初稿、修改稿还是算草或者草图。这正反映出他一贯的工作作风，至今他的所有手迹都保持着这种一丝不苟、严肃认真的精神。从他的算稿，读者可以看到那一串串排列整齐的数据，有的长达八位。要知道在那时最好的计算工具是手摇的机械式计算器，而连最简单的对数函数和三角函数都要从厚厚的专门手册查找，并作内插计算才能得到。可见，这些数据后面包含了多少辛勤繁杂、严密细致的劳动。

钱学森的博士论文是属于流体力学方面的，但是在学位论文工作之后，他却首先转而研究薄壁扁壳和薄壁圆柱壳的失稳问题。这是因为这些都是当时困扰着航空工程师们的难题。那时的实验结果和理论推算之间存在着很大的差别，这是经典线性理论所不能解释的，对这种十分困难的非线性现象，那时还没有相应

的理论。在薄壁圆柱壳的失稳问题上，他十分认真地观察和分析了实验结果，经过反复尝试，他认识到，在失稳前后有一个能量跳跃过程，并且发现了一种符合实验现象的模态，用它可以得到远比线性理论所得结果好得多的失稳临界载荷。这个结果是在他经历了多次失败后才取得的，仅现在收集到的有关这一问题的手稿就有800多页，而正式发表的论文却只有10页。难怪在完成这项研究时，他在存放手稿的信袋上用红笔写下了“Final”，即“最后的定稿”。但是作为一名严肃的科学家，他意识到该理论仍有不足之处，因此他又写下了“Nothing is final”，即“(科学上)没有什么认识是最后的”这几个醒目的字。

这项工作典型地反映了钱学森当时研究工作的一个特点，这就是在复杂的现象中努力抓住最本质的东西，并在此基础上建立数学模型。由于求解非线性微分方程通常是极其困难的，因此在那时必需借助于进一步的近似才能得到解答，正因为如此，这个解答的正确性还需要经过实验的验证才能得到确认。到了今天，由于高速电子计算机的发展和数学理论的进展，人们在企图反映事物的主要矛盾和求解非线性方程方面，客观上所受的制约少多了，可以说发生了质的变化。对于一大批理论上成熟的问题，完全可以借助于计算机解决问题，而无需做多余的人为假设。在帮助认识复杂现象的内在规律方面，计算机作为数值实验的工具，也同样起很大的作用，因此在研究方法上，计算机把人们带入了一个新时代，人们对此必须有充分的认识。在这方面，钱学森也一直是个积极的倡导者。

出于以上原因，他的论文手稿有另一个突出的特点，那就是，在数学公式推导之后，必然有数字演算，以表明理论结果不仅逻辑上站得住，而且数值上也与实验结果或实际经验相符，以表明理论公式是可靠的。完成这一过程往往需要多次反复，工作是十分艰辛的。

他十分强调，研究工作必须建立在对客观现象做认真的观察和前人工作的基础上，必要的话还需要自己做实验。《手稿》为表现这一点，特意从他的手稿中选出了关于文献调研的一份材料。他还认为仅仅知道在哪里可以找到所需要的资料，是远远不够的，必须切实消化和掌握它们，变成刻记在自己脑海中可以反复思考、随时调用和加工的东西。

他还指出，对于一个复杂的问题，往往需要经过多次的反复，经历若干个认识和再认识的过程，才能得到正确的结论。

引导这些反复的是随时将阶段性结论与实验结果或实践经验的对比。只有当通过了这些考验，而且在逻辑上又是严密的时候，才能肯定这个结论。

《手稿》还包含反映钱学森与同事和同行交流的材料。钱学森十分重视学术交流和不同观点的交锋。他不仅主持自己办的讨论班，而且经常参加别的讨论班，把它们作为自己教学和科研工作的一部分。他也很重视个人间的交流，并时常旁听一些感兴趣的课程，以丰富自己的知识。这是他能大跨度地转移他的研究领域并迅速取得成果的重要原因之一。

综观《手稿》的全部内容，读者可以从中看到，钱学森早年在美国从事航空航天领域及其相关学科的理论研究和风洞等问题的工程设计，为他回到祖国，在技术上领导我国火箭和航天事业奠定了广泛而坚实的基础，而且为开辟更广阔的技术科学领域做好了充分准备。

四

最后，我们要特别指出，钱学森在当年那种非常情况下急切回国之后，他的这些珍贵手稿曾散落在他的办公室和实验室的各个角落里。是他的好友，中国人民的朋友 Frank E. Marble（弗朗克・马勃）教授将它们一一收集起来，并加以初步分类和整理，在重开中美两国人民友好的新时期，将它们送回到钱学森的祖国。1996 年 12 月 6 日，在中国科学院力学研究所举行了手稿交接仪式，国防科工委科技委秘书长王寿云少将代表国防科工委，力学研究所所长薛明伦研究员代表力学所，接收钱学森的手稿，他们对 F. Marble 教授这一友好的举措表示衷心的感谢。

正如 F. Marble 教授所说，能如此完整地收集到一位杰出科学家长达 20 年连续不断的科研工作手稿是十分难得的。为此，我们也要向他表示深切的感谢。我们认为他为我国科技文献档案提供了一份极其难得、异常珍贵的材料，它不仅有很高的收藏价值，而且对一代代年轻学子有直接的教育意义。由于篇幅所限，尽管《手稿》不能包涵钱学森手稿的全部科学内容，但它确实是钱学森的科学精神、科学态度和科学作风的典型代表。因此，我们把它献给读者，希望在一定程度上也能起相同的作用。《手稿》的出版也实现了

F. Marble 教授认为理应要把这些珍贵材料归还给钱学森所热爱、所奉献的祖国的宿愿。

还应当说明：由于编者水平所限，《手稿》一定存在缺点和错误，欢迎读者批评和指正。

F. Marble 教授将他多年收集的钱学森手稿送回我国的倡议，得到了国防科学技术工业委员会和中国科学院领导的大力支持，手稿分两批运回中国。在首次展出这些手稿时，师昌绪院士倡议将手稿编辑出版。朱兆祥教授及时提供了钱学森回国后不久参观访问东北时发表的演讲和观感以及酝酿和筹建力学研究所的有关记录，对了解钱学森倡导技术科学的思想以及钱学森为推进新中国的技术科学和航天技术的发展的宏图大略帮助极大。

本《手稿》的另一位积极倡导者、支持者是王寿云同志。他曾经长期担任钱学森的秘书，本人又是从北京大学数学力学系毕业。在钱老的指导下，他曾对钱老在国外的这一段工作进行清理和分析研究，提出过一些深刻的见解。以他为主要撰稿人，为《中国现代科学家传记》第一集所撰写的“钱学森”条目中对钱老早年工作的评价是我们编辑整理本《手稿》的主要参考依据之一。但是，就在本书选材、整理和编辑加工过程中，王寿云同志不幸去世，这对本书来说，无疑是个重大损失。本书的出版也是对王寿云同志的告慰。

山西教育出版社社长任兆文对本书的出版十分关心并亲自过问。出版社的金山同志出于对钱学森的崇敬之情，为出版本《手稿》做出了奉献。何善堉研究员对工程控制论部分的选材提出了初步方案。戴汝为院士审阅了前言，并对有关工程控制论部分的内容，提出了宝贵的修改意见。原国防科学技术工业委员会和中国科学院力学研究所的有关工作人员对手稿的整理和《手稿》的出版做出了一定的贡献。我们在此一并向他们表示诚挚的感谢。

手稿交接仪式上给 Frank E. marble 教授的感谢信

LETTER OF THANKS

Dear Professor & Mrs. Frank E. Marble:

We feel that we must not let the occasion pass without writing down in words about how much we appreciate your unrelenting efforts in the searching, sorting, marking and classification, and safe keeping of Professor Qian Xuesen's manuscripts over so many years, and to thank you for bringing us in person the last batch of his manuscripts.

As stated in your letter to the Xi'an Jiaotong University at the inauguration ceremony of the Qian Xuesen Library last April, it has long been your conviction that Qian's manuscripts should be returned to his own country. Today we are extremely happy to witness that your wish has finally been admirably fulfilled.

Part of Qian's manuscripts have been displayed at several exhibitions both in Xi'an and in Beijing. These exhibitions were very well received, because the manuscripts show a devoted and highly disciplined eminent scientific mind at work, and are therefore of great educational value. We firmly believe that these manuscripts will form part of Qian's legacy to his homeland, and will be a constant source of inspiration for students of science and engineering.

Here lies the great value of your contribution, for which we are most grateful.

Once again, our warmest thanks and best wishes.

Science and Technology Committee
Commission of Science, Technology
and Industry for National Defence
People's Republic of China

Institute of Mechanics
Chinese Academy of Sciences

December 5, 1996

Frank E. Marble 教授在西安交通大学钱学森图书馆开幕式上的书面发言

COMMENTS FOR THE DEDICATION OF THE LIBRARY AT THE XI'AN JIAOTONG UNIVERSITY, DECEMBER 11, 1995

It is a pleasure and an honor to participate in the dedication of this fine library in the mame of our friend, the eminent scientist and engineer, Tsien Hsue – shen.

Perhaps a few words about my small contribution to this celebration are appropriate. Although Tsien and I met briefly in 1946, it was the years of 1949 through 1955, after he returned ro The California Institute of Technology, that I had the privilege of working closely with him. It was a period of enormous producrivity for him and an intensely stimulating experience for me. During this period, and I am convinced that it was always so, Tsien exhibited skilfully, with keen knowledge of important technical issues and with the remarkable physical and analytical insight that characterizes all of his work. With these choices made, he moved quickly to the solution of the problem.

As an integral part of this process. Tsien wrore as he thought – producing an accurate. timely history of the initiation, analysis, and growth of each problem – on to its completion. Once the fimal document was polished to his satisfaction, these notes and calculations went into a large envelope (usually brown!) which. as I witnessed several times, was tossed casually onto his book shelves.

Many other matters occupied Tsien' s mind as he rerurned to China and this material remained on his office shelves. Because I was probably the only one who knew the sigificance of their contents, I gathered these envelopes into files for safe keeping. Gradually, during the following years, I came across other files scattered in remote storage areas about the.

Guggenheim Aeronautical Laboratory. To my surprise I found similar collections of notes, in similar brown envelopes, that dated back to his earliest work with Professor von Karman.

Among these was the first theoretical analysis and calculations of boundary layers in supersonic flow, which constituted the first chapter of his doctoral thesis! Even a casual study of this 1937 work shows that he obtained results that were far broader than those pubished. In the end I was satisfied that I had assembled ass, or very nearly all, of Tsien' s scientific and technical records from the years 1936 – 1955 when he worked in the United States. It was always my aim that this material should be returned to his home and to my great personal satisfacrion, through the kind efforts of Cheng Che – min, this was accomplished about two years ago.

I believe that these datailed research notes, when correlated with his published works, constitute an unusually valuable record of the early years of a most remarkable scietific mind. I doubt that there is a comparable collection recording the vitally important technological contributions of any contemporary scientist and engineer. It is completely fitting and appropriate that this marerial should find its permanent home in the university at which their author recieved the foundations for his work.

Finally, I should like to take this opportunity of sending warm greetings from myself and from Ora Lee Marble to our dear friends Tsien Hsue – shen and Tsiang Yin with whom we shared treasured experiences during their years in Pasadena.



钱学森简历

钱学森，1911年12月11日出生于上海，是独生子。父亲钱均夫（原名家治，号均夫）是浙江杭州一没落丝商第二子，少小就学于当时维新的杭州求是书院，曾到日本学教育和地理、历史。母亲章兰娟是当时杭州富商的女儿。钱学森的外祖父欣赏钱均夫的才华，把自己的女儿许配给他。民国成立后，钱均夫就职北京当时的教育部。钱学森在3岁时随父亲到了北京，上过蒙养院（幼儿园）、女师大附小、师大附小和师大附中。

在北京师大附中时，对钱学森影响最深的几位老师是：林砺儒、王鹤清、董鲁安（于力）以及几何老师傅种孙、生物老师愈谟（俞君适）、博物老师李士博和美术老师高希舜（后来是著名国画大师）。林砺儒是校长（当时称主任），他制定了一套以启发学生智力为目标的教学方案。王鹤清是化学老师。他给钱学森自由到化学实验室做实验的便利，这启发了他对科学的兴趣。董鲁安是国文老师，在课堂上常常用较长的时间讨论时事，表示厌恶北洋军阀政府，憧憬国民革命军北上（后来他去了解放区）。他的教学使钱学森产生对旧社会腐败的深切不满和对祖国前途、人民命运的无比关心。钱学森一次在图书馆借了一本讲相对论的小册子，书中第一句话提到20世纪有两位大师：一位是自然科学大师A. 爱因斯坦(Einstein)，一位是社会科学大师列宁。钱学森当时对列宁这位大师还不甚了解。傅种孙那时已是师大数学讲师，在中学课堂上把道理讲得很透。钱学森后来认为，在初中三年级听傅老师的几何课，使他第一次得知什么是严谨的科学。钱学森对老师们的教诲感激不尽，他后来说："我若能为国家为人民做点事，也与中小学老师的教育不可分！"

1929年中学毕业后，钱学森为复兴祖国，决心学工科，考入上海交通大学机械工程系。当时上海交大专重考试分数，学期终了平均分数算到小数点以后两位，大家都为分数而奋斗。初入交大的钱学森对这里的"分数战"虽不甚满意，但也不甘落后，非考90分以上不可。在交大，钱学森非常感激两位倡导把严密的科学理论与工程实际结合起来的老师，一位是工程热力学教授陈石英，一位是电机工程教授钟兆琳。

1930年暑假后期，钱学森得了伤寒病，在杭州家里卧病一月余，

后因体弱休学一年。在这一年里，他第一次接触到科学的社会主义。钱学森爱好美术，在书店买了一本讲艺术史的书，不曾想这本书是一位匈牙利社会科学家用唯物史观的论点写的。他从未想到对艺术可以进行科学分析，所以对这一理论发生了莫大的兴趣。接着他读了普列汉诺夫的艺术论、布哈林的唯物论等书，又看了一些西洋哲学史，也看了胡适的《中国哲学史大纲》(上册)。读了这么多书，他感到只有唯物史观和辩证唯物主义才是有道理的，唯心主义等等没有道理；经济学也是马克思的有道理，而资产阶级经济学那一套理论，则不能自圆其说。休学期满回到学校，钱学森开始接触到共产党的外围组织，参加过多次小型讨论会，从那里他知道了红军和解放区的存在。小组的领导人乔魁贤，是当时交大数学系的学生，小组还有许邦和、袁轶群和褚应璜。后来乔魁贤被学校开除；钱学森和小组的联系也逐渐中断，仍埋头读书，每学期平均分数都超过 90 分，因而得到免交学费的奖励。在上海交大，好友有林津、熊大纪、郑世芬、罗沛霖、茅于恭等。假期在杭州，因与学音乐的表弟李元庆思想相投而常交往，从他那里略闻左翼文艺运动的情况。

在 1934 年暑假，钱学森从上海交大机械工程系铁道机械工程专业毕业。尚未派定工作，就考取了清华大学公费留学，专业是飞机设计，两位导师一位是王助，一位是王士倬。王助是我国早年航空工程师，设计制造了中国第一代飞机，他教导钱学森重视工程技术实践和制造工艺问题。王士倬是清华教授。依照清华关于留美学生的规定，钱学森在 1934—1935 年到杭州笕桥飞机厂实习，又到南京、南昌空军飞机修理厂见习，最后到北京参观清华并拜访导师王士倬，也见到王士倬当时的助教张捷迁。

1935 年 8 月，钱学森从上海坐美国邮船公司的船离国，同船的留美同学有徐芝纶、夏勤铎等。当时钱学森的心情是：中国混乱、豺狼当道，暂时到美国去学些技术，他日回来为国效劳。到了美国入麻省理工学院航空系，成绩不但比美国学生好，而且比同班的其他外国人都好，这使他感到作为一个中国人而自豪。因为学工程一定要到工厂去，而当时美国航空工厂不欢迎中国人，所以一年后他开始转向航空工程理论，即应用力学的学习。于是决定追随当时在加利福尼亚理工学院（简称加州理工学院）的力学大师 T. 冯·卡门 (von Kármán) 教授。1936 年 10 月，钱学森转学到加州理工学院，开始了与冯·卡门教授先是师生后是亲密合作者的情谊。冯·卡门第一次见到钱学森时，看到的是一位个子不高、仪表严肃的年轻人；他异常准确地回答了教授的所有提问；他的思维敏捷和富于智

慧，顿时给冯·卡门以深刻的印象。冯·卡门教授教给钱学森从工程实践提取理论研究对象的原则，也教给他如何把理论应用到工程实践中去。冯·卡门每周主持一次研究讨论会(research conference)和一次学术研讨会(seminar)，这些学术活动给钱学森提供了锻炼创造性思维的良好机会。

到加州理工学院的第二年，即1937年秋，钱学森认识了热心研究火箭技术的同学F.J.马林纳(Malina)，共同具有的火箭、音乐和政治兴趣，使两位青年结成良友。由马林纳介绍，钱学森参加了当时加州理工学院的马列主义学习小组，也得识该小组的书记、化学物理助理研究员S.威因鲍姆(Weinbaum)。小组曾念过英国J.S.L.斯崔奇(Strachey)著的一本书，后来也学习过恩格斯的《反杜林论》；每星期例会常讨论时事，主题是反法西斯和人民阵线；小组还参加过美国共产党书记E.白劳德(Browder)的几次讲演会。1938年冬，第二次世界大战爆发后，不少小组成员加入了美国共产党，也有人参加了军事研究，这个小组就无形解散了。后来，马林纳在麦卡锡(Joseph R. McCarthy)主义反动浪潮席卷美国的初期，辞去了加州理工学院的喷气推进实验室主任职务，去巴黎为联合国教科文组织服务，并成为现代派画家，1981年11月9日在巴黎病逝。

钱学森在加州理工学院的博士论文工作是在1939年6月结束的，论文为《高速气动力学问题的研究》等四篇。取得航空、数学博士学位后，任加州理工学院航空系的助理研究员，一直到1944年。在这一段时间内，先从事薄壳体稳定性的研究，1940年完成了研究课题，并撰写了论文在美航空学会年会上宣读，算是独立研究，出了师。此后钱学森成为冯·卡门的助手、帮助他指导研究生的论文。1940年，由于王助的推荐，钱学森成为成都航空研究所的通讯研究员，写了一篇题为《高速气流突变之测定》的专论(刊登在该所报告第二号)。

1941年，从加拿大来了几位庚子赔款的留学生：郭永怀、林家翘、傅承义，1942年又来了钱伟长。钱学森和他们相处得比较密切，一般是一起吃晚饭，并常常讨论各种问题。钱伟长多才多艺，傅承义专攻地球物理。钱学森和郭永怀最相知(后来在1957年初，有关方面询问谁是承担核武器爆炸力学工作最合适的人选时，钱学森毫不迟疑地推荐了郭永怀)。1943年秋冬，周培源也到加州理工学院来做研究工作，找冯·卡门教授讨论湍流统计理论等。这一群中国同学，还有张捷迁、毕德显，星期天总到周培源老师家去玩，高谈国事，也

替师母王蒂瀓烹制午晚餐。

到1942年，钱学森的研究工作已有了成绩，并教了些学生；同时由于美国战时军事科学研究的需要，暂时放松了对外国人的限制，故得以参加机密性工作。1939年前后，美国空军开始支持火箭研究。1942年，美国军方委托加州理工学院举办喷气技术训练班，钱学森是教员之一，与陆海空三军技术人员有了接触。后来美军从事火箭导弹的军官中，有不少是他当时的学生。1944年，美国陆军得知德国研制V-2火箭的情报，遂委托冯·卡门教授领导，马林纳为副，大力研究远程火箭。美军原始型的"下士"式导弹就是他们那时开始设计的。钱学森负责理论组，把林家翘、钱伟长也请了来，进行弹道分析、燃烧室热传导、燃烧理论研究等工作。同时钱学森还当了航空喷气公司(Aerojet Company)的技术顾问，加州理工学院提升钱学森为讲师。冯·卡门对钱学森是很欣赏的，所以在1945年初他被空军聘为科学咨询团团长的时候，提名钱学森为团员。这个团为美国空军提供了一个远景发展意见，钱学森从中学到从大处和远处设想科技发展问题的方法。1945年5月，第二次世界大战结束的前夕，钱学森随科学咨询团去欧洲，考察英、德、法等国的航空研究，特别是法西斯德国的火箭技术发展情况。这时加州理工学院提升他为副教授。这一时期，他取得了在近代力学和喷气推进的科学研究方面的宝贵经验，成为当时有名望的优秀科学家。冯·卡门这样评价钱学森："他在许多数学问题上和我一起工作。我发现他非常富有想象力，他具有天赋的数学才智，能成功地把它与准确洞察自然现象中的物理图象的非凡能力结合在一起。作为一个青年学生，他帮我提炼了我自己的某些思想，使一些很艰深的命题变得豁然开朗。"

1946年暑期，冯·卡门教授因与加州理工学院当局有分歧而辞职，作为冯·卡门的学生，钱学森也离开加州理工学院，再到麻省理工学院任副教授，专教空气动力学专业的研究生。1947年初，36岁的钱学森进入了麻省理工学院年轻的正教授行列。同年夏季，钱学森向麻省理工学院当局请假回国探亲，9月中和蒋英结婚。蒋英是蒋百里、蒋左梅夫妇的第三女，生于1920年9月，是在维也纳和柏林受过良好的音乐教育的女高音声乐家。蒋百里是旧中国著名的军事理论家，蒋左梅是日裔友人。

1948年祖国解放事业胜利在望，钱学森开始准备归国。为此他要求退出美国空军科学咨询团，但直到1949年才得以实现。他兼任的美国海军炮火研究所顾问的职务，直到1949年秋从麻省理工学

院回到加州理工学院就任喷气推进技术教授职务时才辞去。

1949年5月20日,钱学森收到美国芝加哥大学金属研究所副教授研究员、留美中国科学工作者协会(简称留美科协)美中区负责人葛庭燧写来的信,同时转来1949年5月14日曹日昌教授(中共党员,当时在香港大学任教)写给钱学森的信,转达即将解放的祖国召唤他返国服务、领导中华人民共和国航空工业建设之切切深情。这时钱学森还看到周培源给林家翘的信,得知北京西郊解放时的良好情况。也见到在加州理工学院当研究生的罗沛霖(曾经以非党技术人员身份在延安工作过),他认为钱学森回国为解放了的祖国服务的时候到了。钱学森遂加紧了回祖国的准备,以便实现他多年的夙愿。

但这时正直麦卡锡主义横行,美国全国掀起一股要雇员们效忠政府的歇斯底里狂热。几乎每天都发生对大学和其他机构进行审查或威胁性审查的事件。加州理工学院也被涉及,因威因鲍姆下狱,怀疑落到钱学森身上。1950年7月,美国政府决定取消钱学森参加机密研究的资格,理由是他与威因鲍姆有朋友关系,并指控钱学森是美国共产党员,非法人境。钱学森这时立即决定以探亲为名回国,准备一去不返。但当他一家将要出发的时候,钱学森被拘留起来,两星期后虽在几位美国同事好友的大力帮助下保释出来,但继续受到移民局根据麦卡锡法案进行的迫害,行动处处受到移民局的限制和联邦调查局特务的监视,被滞留5年之久。1955年6月的一天,钱学森夫妇摆脱特务监视,在一封写在一张小香烟纸上寄给在比利时亲戚的家书中,夹带了给陈叔通先生的信,请求祖国帮助他早日回国。陈叔通先生收到这封信的当天,就把它送到周恩来总理手里。1955年8月1日、中美大使级会谈在日内瓦开始,王炳南大使按照周总理的授意,以钱学森这封信为依据,与美方进行交涉和斗争,迫使美国政府不得不允许钱学森离美回国。8月5日,钱学森接到美国政府的通知,说他可以回国。但在乘坐美国邮船的归途中,他仍被当作犯人对待。

在1950年到1955年这一段争取回国的时间里,钱学森因受到特务监视,感到压力很大,除了教书和做研究工作以外,学术活动和社会活动参加得很少,但仍未放弃学术研究。钱学森这个时期的主要开创性的研究成果是1954年在美国发表的《工程控制论》一书及讲授力学工作介质物理性质的理论“物理力学”。当钱学森在回国前夕同蒋英带着幼儿钱永刚、幼女钱永真向他的老师告别时,冯·卡门充满感情地说:“你现在学术上已经超过我!”

就在美国政府迫害钱学森的5年中，加州理工学院的许多美国朋友安慰他，千方百计地给他解决困难，表示了真诚的友情，如W. R. 西尔斯（Sears）教授、F. 马布尔（Marble）教授、M. 米尔斯（Mills）、登肯·兰尼（Duncan Rannie）等。

钱学森后来回顾在美国的经历时说："我从1935年去美国，1955年回国，在美国待了20年。20年中，前3—4年是学习，后十几年是工作，所有这一切都在做准备，为了回到祖国后能为人民做点事。我在美国那么长时间，从来没想过这一辈子要在那里待下去。我这么说是有根据的。因为在美国，一个人参加工作，总要把他的一部分收入存入保险公司，以备晚年退休之后用。在美国期间，有人好几次问我存了保险金没有，我说一块美元也不存，他们感到很奇怪。其实没什么奇怪的。因为我是中国人，根本不打算在美国住一辈子。"

钱学森一家1955年10月8日到达香港，同日过国境，回到了祖国。从香港上码头开始，通过与中国旅行社同志的接触，感受到了祖国的温暖。进入国境，钱学森一家见到了科学院派来接他们的朱兆祥。党和政府对他们的照顾无微不至。钱学森受到广东省委书记陶铸的接见并在广州参观。经过上海、杭州，最后到了北京。不久，领导上安排钱学森到东北去参观、看了农村和工厂，特别是飞机厂等，饱览了祖国欣欣向荣的景象。

1955年11月，钱学森和钱伟长合作筹建中国科学院力学研究所。1956年1月5日，力学所正式成立，钱学森任第一任所长，直至70年代后期。在钱学森倡议下，中国力学学会在1957年正式成立，钱学森被一致推举为第一任理事长。1958年任中国科学技术大学近代力学系主任，讲授星际航行概论和物理力学。

1956年春，钱学森应邀出席中国人民政治协商会议第二届全国委员会第二次全体会议，并在会上发言。2月1日晚，毛泽东主席设宴招待全体委员，并特别安排钱学森同自己坐在一起，进行了亲切的谈话。这是一个有意义的时刻，它表明了钱学森从1955年10月8日回到祖国后，已全身心地投入一项新的事业——中国共产党领导的现代化建设事业。1959年经杜润生、杨刚毅介绍，钱学森加入了中国共产党。

1957年，钱学森所著《工程控制论》获中国科学院自然科学奖一等奖，并被补选为中国科学院学部委员。同年9月，国际自动控制联合会（IFAC）成立大会推举钱学森为第一届IFAC理事会常务理事。1961年，在中国自动化学会成立大会上，全体代表一致推举钱学森

为首任理事长。

在40年代试验导弹的日子里，钱学森就意识到导弹日益增长着的重要性，需要一种他称之为喷气式武器部的新机构，用新的军事思想和方法专门进行研究。新中国国防建设的需要，为他实现这一预见提供了历史的机遇。在哈尔滨参观中国人民解放军军事工程学院时，院长陈赓大将专程从北京赶回哈尔滨接见钱学森，他问钱学森的第一句话是："中国人搞导弹行不行?"钱学森说："外国人能干的，中国人为什么不能干?"陈赓大将说："好！就要你这一句话。"这次谈话，决定了钱学森从事火箭、导弹和航天事业的生涯。1955年12月27日，万毅根据彭德怀元帅的指示，详细地听取了钱学森关于如何发展我国火箭导弹技术的意见。1956年2月17日，在周恩来总理的鼓励下，作为一个刚刚回归祖国不久的科学家，钱学森怀着对新中国国防事业强烈的责任感，给国务院写了关于《建立我国国防航空工业的意见书》(当时为保密起见，用"国防航空工业"这个词来代表火箭、导弹和后来所称的航空航天技术)。《意见书》指出："健全的航空工业，除了制造工厂之外，还应该有一个强大的为设计服务的研究及试验单位，应该有一个作长远及基本研究的单位。自然，这几个部门应该有一个统一领导的机构，做全面规划及安排的工作。"《意见书》提出了我国"国防航空工业"的组织草案、发展计划和具体步骤，并且开列了一张可以调来做高级技术工作的21人名单，包括任新民、罗沛霖、梁守槃、胡海昌、庄逢甘、罗时钧、林同骥等。《意见书》立即引起中央的重视，周恩来总理在1956年3月14日亲自主持会议研究，决定由周恩来总理、聂荣臻元帅和钱学森等筹备组建导弹航空科学研究的领导机构——航空工业委员会，委员会下设立：(1)设计机构；(2)科学机构；(3)生产机构。1956年4月13日，国务院成立了以聂荣臻元帅为主任的航空工业委员会(当时对外不公开)，钱学森被任命为委员。

1956年春，周恩来总理亲自领导数百名科学技术专家，制订新中国第一个远大的规划——《1956至1967年科学技术发展远景规划纲要》，确定了57项国家重要科学技术任务。由钱学森主持，在王弼、沈元、任新民等的合作下完成了第37项(《喷气和火箭技术的建立》)的规划。钱学森等在这项重要科学技术任务的说明书中指出："喷气和火箭技术是现代国防事业的两个主要方面：一方面是喷气式的飞机，一方面是导弹。没有这两种技术，就没有现代的航空，就没有现代的国防。建立了喷气和导弹的技术，民用航空方面的科学技

术问题也就不难解决”;“本任务的预期结果是建立并发展喷气和火箭技术，以便在12年内使我国喷气和火箭技术走上独立发展的道路并接近世界先进的科学技术水平,以满足国防的需要”;解决本任务的途径:“必须尽快建立包括研究、设计和试制的综合性的导弹研究机构,并逐步建立飞机方面的各个研究机构”;解决本任务的大体进度:“1963—1967年,在本国研究工作的指导下,独立进行设计和制造国防上需要的、达到当时先进性能指标的导弹”;组织措施是:“在国防部的航空委员会下成立导弹研究院,该院自1956年起开始建设,1960年建成”。1956年5月10日,聂荣臻元帅提出《关于建立我国导弹研究工作的初步意见》,并且建议:在航空工业委员会下设立导弹管理局，钱学森任总工程师；建立导弹研究院，钱学森任院长。钱学森很快受命负责组建我国第一个火箭、导弹研究院——国防部第五研究院。1956年10月8日，恰好是钱学森回归祖国一周年的日子,聂荣臻元帅亲自主持五院成立仪式。这一天也是对新中国156名大学毕业生进行导弹专业教育训练班的开课纪念日。钱学森主讲《导弹概论》。在1942年加州理工学院喷气技术训练班授课14年之后,钱学森为能在自己的国家培养我国第一批火箭、导弹技术人才,感到无比激动。这批受训的大学生,后来成为我国火箭、导弹与航天技术队伍的骨干。1957年2月18日，周恩来总理签署国务院命令,任命钱学森为国防部第五研究院第一任院长。从此,在周恩来总理、聂荣臻元帅直接领导下,钱学森开始了作为新中国火箭、导弹和航天事业技术领导人的长期经历。1957年11月16日,周恩来总理任命钱学森兼任国防部第五研究院一分院院长。1958年5月29日,聂荣臻元帅同黄克诚、钱学森一起部署了我国第一枚近程导弹的制造工作。1960年11月5日，在聂荣臻元帅现场亲自指导下，以张爱萍将军为主任，孙继先、钱学森、王诤为副主任的试验委员会,在我国酒泉发射场成功地组织了我国制造的第一枚近程导弹的飞行试验。正如聂荣臻元帅在庆祝宴会的祝酒词中所说,在祖国的地平线上，飞起了我国自己制造的第一枚导弹，这是我国军事装备史上一个重要的转折点。1964年6月29日，我国第一个自行设计的中近程导弹进行飞行试验获得成功。1966年10月27日,遵照周恩来总理“严肃认真、周到细致、稳妥可靠、万无一失”的指示，钱学森协助聂荣臻元帅,在酒泉发射场直接领导了用中近程导弹运载原子弹的“两弹结合”飞行试验，导弹飞行正常，原子弹在预定的距离和高度实现核爆炸。这次史无前例的试验标志着中国开始有了用

于自卫的导弹核武器，也标志着《1956至1967年科学技术发展远景规划纲要》规定的“1963—1967年在本国研究工作的指导下，独立进行设计和制造国防上需要的、达到当时先进性能指标的导弹”这一任务的提前完成。第二天，即1966年10月28日，《纽约时报》用这样的文字报道了这一重大事件：“一位15年中在美国接受教育、培养、鼓励并成为科学名流的人，负责了这项试验，这是对冷战历史的嘲弄。1950—1955年的5年中，美国政府成为这位科学家的迫害者，将他视为异己的共产党分子予以拘捕，并试图改变他的思想，违背他的意愿滞留他，最后才放逐他出境回到自己的祖国。

早在1953年，钱学森就研究了星际航行理论的可行性。1958年，中国科学院成立以钱学森为组长、赵九章和卫一清为副组长的领导小组，负责筹建人造卫星、运载火箭以及卫星探测仪器和空间物理的设计、研究。1961年6月，在钱学森、赵九章等的倡导下，中国科学院开始举办了持续12次的星际航行座谈会，钱学森在第一次座谈会上发表了题为《今天苏联及美国星际航行火箭动力及其展望》的讲演。1963年，中国科学院成立了由竺可桢、裴丽生、钱学森、赵九章领导的星际航行委员会，负责组织制订星际航行发展规划，安排预先研究课题。1965年1月8日，钱学森正式向国家提出报告，建议早日制订我国人造卫星的研究计划并列入国家任务。钱学森指出：“自从苏联在1957年10月4日发射第一颗人造地球卫星以来，中国科学院及原第五研究院对这项新技术就有些考虑，但未作为研制任务。现在看来，人造卫星有以下几种已经明确的用途：测地卫星、通讯及广播卫星、预警卫星、气象卫星、导航卫星、侦察卫星。重量更大的载人卫星在国际上的应用，现在虽然还不十分明确，也得有所准备。现在我国弹道式导弹已有一定的基础，现有型号进一步发展，即能发射100公斤左右重量的仪器卫星。这些工作是复杂艰巨的，必须及早开展有关的研究、研制工作，才能到时拿出东西。因此，建议国家早日制订我国人造卫星的研究计划，列入国家任务，促进这项重大的国防科学技术的发展。”聂荣臻元帅很重视钱学森的建议，指出：“只要力量上有可能，就要积极去搞。”1965年4月29日，国防科委向中央专门委员会报告了邀请张劲夫、钱学森、孙俊人及国家科委、国防工办专业局的负责同志和专家进行研究的结果，提出了在1970年或1971年发射我国重量为100公斤左右的第一颗人造地球卫星的设想。中央专门委员会于1965年5月4、5日召开的第12次会议和8月9、

日、10日召开的第13次会议，原则批准了我国第一颗人造卫星的规划方案，以及争取在1970年左右发射我国第一颗人造卫星的设想。钱学森为解决人造卫星研制中的许多关键技术问题贡献了智慧。譬如，在1966年6月下旬，第一颗人造卫星的运载火箭"长征一号"，为解决滑行段喷管控制问题而进行的滑行段晃动半实物仿真试验，出现了晃动幅值达几十米的异常现象。钱学森亲临现场，在讨论中认定：此现象在近于失重状态下产生，原晃动模型已不成立，此时流体已呈粉末状态，晃动力很小，不影响飞行。后来多次飞行试验证明，这个结论是正确的。在"文化大革命"的日子里，钱学森协助周恩来总理，为领导人造卫星研制计划的正常进行，发挥了特殊的作用，譬如，由于"文化大革命"，"长征一号"运载火箭试车无法进行，1969年7月17日、18日、19日和25日，周恩来总理连续4次召开会议，解决二级和三级地面试车问题，委派钱学森协同七机部军管会副主任杨国宇全权处理有关试车事宜。从而得以在8月22日取得试车成功。1970年，在周恩来总理的直接关怀下，钱学森、李福泽、杨国宇、任新民、戚发轫等在酒泉卫星发射场组织实施了第一颗人造卫星的发射工作。1970年4月24日，重量为173公斤的我国第一颗人造卫星发射成功。钱学森和发射基地的领导人及试验队的代表在现场发表了热情洋溢的讲话。"五一"国际劳动节晚上，毛泽东主席、周恩来总理在天安门城楼上接见了钱学森、任新民等参加第一颗卫星工程研制的代表。这颗卫星向全世界播送的《东方红》乐曲，宣告了新中国迎来了航天时代的黎明。

周恩来总理和聂荣臻元帅是钱学森最崇敬的我国科技事业领导人。他说过："按照我的体会，周总理、聂老总就是把他们过去在解放战争中组织大规模作战的那套办法，有效地用到科技工作中来，把成千上万的科技大军组织起来了。"

钱学森1965年2月15日任第七机械工业部副部长，1968年兼任中国空间研究院第一任院长，1970年6月12日任国防科学技术委员会副主任，1982年任国防科学技术工业委员会科学技术委员会副主任(1987年7月任高级顾问)。钱学森是中国共产党第九、第十、第十一、第十二届全国代表大会代表和中央委员会候补委员。

1979年，钱学森荣获加州理工学院"杰出校友奖"(The Distinguished Alumni Award)。

1985年，钱学森因对我国战略导弹技术的贡献，作为第一获奖人和屠守锷、姚桐斌、郝复俭、梁思礼、庄逢甘、李绪鄂等获全国科技进步特等奖。

1986年4月—1998年3月，任中国人民政治协商会议第六、七、八届全国委员会副主席；1986年6月—1991年5月任第三届中国科学技术协会主席，现为名誉主席。

1989年6月29日，在美国纽约召开的1989年国际技术与技术交流大会授予钱学森"威拉德 W. F. 小罗克韦尔（Rockwell, Jr.）奖章"和"世界级科学与工程名人"、"国际理工研究所名誉成员"的称号，表彰他对火箭导弹技术、航天技术和系统工程理论做出的重大开拓性贡献，称他"在作为加州理工学院学生时代，冯·卡门教授就因他在喷气推进和超声速飞机设计方面的才智而对他特别宠爱。在有关火箭设计的研究工作中，为发展喷气推进，他引入了钱学森公式。钱学森长期担任中国先驱的火箭和航天计划的技术领导人。他对航天技术、系统科学和系统工程做出了巨大的和开拓性的贡献。"1991年10月国务院、中央军委授予钱学森"国家杰出贡献科学家"荣誉称号和全军一级英模奖章，1999年9月中共中央、国务院、中央军委授予钱学森"两弹一星功勋奖章"，以表彰他对我国科学技术事业，特别是"两弹一星"事业做出的贡献。


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应用力学

1.1
空气动力学

1.1.1
Boundary Layer in Compressible Fluids
可压缩流体边界层

1934年，钱学森从交通大学机械工程系铁道机械工程专业毕业，有志于发展中国的航空事业，考取了公费留美，学习飞机设计。1935年，进入美国麻省理工学院航空系，一年后获硕士学位。就在这时，他了解到美国的航空工厂不欢迎中国人后，于1936年10月转学到加州理工学院，师从Theodore von Kármán（冯·卡门）学习航空工程理论，即应用力学。

当时航空界正在从低速飞行向高速飞行的方向发展。当飞行速度提高以后，飞行体所受空气的阻力和热效应究竟会发生什么变化？导师von Kármán建议作者对这一问题进行理论探讨。在那时的科技文献中，普遍认为在超声速飞行中空气阻力主要来自击波阻力，而表面摩擦并不重要。至于热效应，一般认为飞行体的表面被周围的空气所冷却。

这一问题在数学求解上的困难在于，高速飞行时飞行体周围的空气密度会发生显著的变化，方程不再是线性的。作者采用了von Mises简化边界

层方程的做法，然后运用逐次近似解法求解非线性方程，取得了成功。他把已知的不可压缩流动的解推广到可压缩流动，即飞行马赫数比较大的情况，得到有关高速飞行体的阻力和表面热效应两方面的重要结论：第一，在高速飞行中，可压缩性对表面摩擦具有重要的影响，高速弹体和火箭因摩擦引起的阻力将超过击波引起的阻力；第二，当飞行马赫数增大到一定数值，飞行体表面的空气薄层中所产生的热不仅不能被忽略，而且将对飞行体起加热的作用。作者这一工作从理论上预见了实现高速飞行将面临的一大障碍，即后人所称谓的“热障”，必须对飞行体表面采取有效的冷却或防热措施，才能实现高速飞行。

上述研究结果成为他博士论文的一部分，并写了题为“Boundary Layer in Compressible Fluids”（可压缩流体边界层）的论文，发表在1938年的《Journal of Aeronautical Sciences》（航空科学学报）上。这里选印的有关材料共计36页，包括3个部分，即：1. 导师 von Kármán 在工作开始前给作者的书面指导，共3页；2. 作者所写的演算手稿（250页）中的最前面5页；以及3. 作者的论文手稿的一部分，共28页，包括两页提要和两页算例的草稿。从第1部分的材料中，可以看出导师 von Kármán 建议作者试试 von Mises 的方法和逐次迭代近似方法。从第2部分的材料中，可以看到作者的严密的工作作风，工作一开始他便收集和阅读参考文献，并且制订了研究计划，然后从基本方程出发寻求解答。第3部分的手稿中有导师 von Kármán 的多处重要修改。

For Mr Tsien!

Differential equation:

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial y}\left(\rho\mu u \frac{\partial u}{\partial y}\right) \qquad (1)$$

Put

$$u = f\left(\frac{y}{\sqrt{x}}\right), \quad \frac{y}{\sqrt{x}} = \eta$$

then:

$$-\frac{1}{2}\eta\frac{du}{d\eta} = \frac{d}{d\eta}\left(\rho\mu u \frac{du}{d\eta}\right) \qquad (2)$$

Iteration method:

$$\text{denote } (\rho\mu u)_{\text{incorrect}} = g(\eta)$$

$$-\frac{1}{2}\eta\frac{du}{d\eta} = \frac{dg}{d\eta}\frac{du}{d\eta} + g\frac{d^2u}{d\eta^2}$$

or

$$\frac{d^2u}{d\eta^2} = -\frac{1}{g}\left(\frac{1}{2}\eta + \frac{dg}{d\eta}\right)\frac{du}{d\eta} \qquad (3)$$

integrated

$$\log\frac{du}{d\eta} = C - \log g - \frac{1}{2}\int_0^\eta \frac{\eta\, d\eta}{g(\eta)}$$

$$\frac{du}{d\eta} = \frac{A}{g_\eta} e^{-\frac{1}{2}\int_0^\eta \frac{\eta\, d\eta}{g(\eta)}}$$

and

$$u = A\int_0^\eta \frac{d\eta}{g(\eta)} e^{-\frac{1}{2}\int_0^\eta \frac{\eta\, d\eta}{g(\eta)}}$$



Calculate u, ρ, μ versus $g(\eta)$ and iterate.

~~$u = 2A\int \frac{\eta\, d\eta}{2g(\eta)\cdot\eta} e^{-\frac{1}{2}\int\frac{\eta\, d\eta}{g(\eta)}} = 2A\int \frac{1}{\eta} e^{-\frac{1}{2}\int\frac{\eta\, d\eta}{g(\eta)}}$~~ $\qquad g(\eta) = B\sqrt{\eta}$

$$\left.\frac{du}{dy}\right)_0 = \left.\frac{du}{dy}\rho u\right)_0 = \left.\frac{du}{d\eta}\frac{1}{x}\rho_0 u\right)_0 = \frac{2A^2}{B^2}\frac{\rho_0}{\sqrt{x}}$$

I)

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \mu \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)$$

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$$

$$\eta = \frac{y}{\sqrt{x}}$$

$$\rho u = F'(\eta) \qquad \frac{\partial}{\partial x} = -\frac{\eta}{2x}\frac{d}{d\eta}$$

$$\rho v = -\frac{1}{2\sqrt{x}}\left(F - \eta F'\right) = \frac{1}{2\sqrt{x}}\left(\eta F' - F\right) \qquad \frac{\partial}{\partial y} = \frac{1}{\sqrt{x}}\frac{d}{d\eta}$$

$$\frac{1}{\rho} = g(\eta)$$

$$\mu = h(\eta)$$

$$-\frac{1}{2}F'(\eta)\,\eta\,\frac{d}{d\eta}(F'g) + \frac{1}{2}(\eta F' - F)\frac{d}{d\eta}(F'g) = \frac{d}{d\eta}\left[h\frac{d}{d\eta}(F'g)\right]$$

II) Try Mises' method!

Oct. 11

for Mr Tsien

$$-\frac{1}{2}\eta \frac{du}{d\eta} = \frac{d}{d\eta}\left[(\rho\mu u)\frac{du}{d\eta}\right] \qquad \eta = \frac{y}{\sqrt{x}}$$

Introduce $\int_0^u \rho\mu u\,du = F(u)$

$$F'(u) = f(u)$$

Then multiply both side by $f(u)$

$$-\frac{1}{2}\eta \frac{du}{d\eta} f(u) = f(u)\frac{d}{d\eta}\left(f(u)\frac{du}{d\eta}\right)$$

or

$$-\frac{1}{2}\eta\frac{dF}{d\eta} = f(u)\frac{d^2F}{d\eta^2} \qquad (2)$$

and putting

$$f(u) = f_0(\eta) = (\rho\mu u)_0 \qquad \text{(Blasius)}$$

$$\frac{1}{\frac{dF}{d\eta}}\frac{d^2F}{d\eta^2} = -\frac{\frac{1}{2}\eta}{f_0(\eta)}$$

$$\frac{dF}{d\eta} = C e^{-\int_0^\eta \frac{\frac{1}{2}\eta\, d\eta}{f_0(\eta)}} \qquad 3)$$

same as before however now

$$F = C\int_0^\eta d\eta\, e^{-\int_0^\eta \frac{\eta\, d\eta}{2f_0}}$$

and for $\eta = \infty$

$$\int_0^U \rho\mu u\,du = C\int_0^\infty d\eta\, e^{-\int_0^\eta \frac{\eta\, d\eta}{2f_0}}$$

$$= \rho_w\mu_w u_w\left(\frac{du}{d\eta}\right)_w = \rho_w\mu_w\left(\frac{du}{dy}\right)_w\sqrt{x} = C$$

$$\tau = C/\rho_w\sqrt{x}$$

Shell: Caliber = 6". Total length = 4 × 6 = 24"

Altitude = (10 km = 10,000) = 32,500'

Temp = −49.4 = 223.7°K

σ = 0.3367

$\frac{\mu}{\mu_0}$ = 2.140

V = 1500 ft/sec.

a = 0.882 × 1130 = 997

V/a = 1.52.

$R_e = \frac{6760 \times 2 \times 1500}{2.14}$ = 7860000 = 7.86 × 10⁶

$C_f = \frac{1.786 \times 10^{-3}}{\sqrt{7.86}}$ = 0.000659

$C_{D_f} = \frac{0.25 \times 0.000659 \times 8}{0.50}$ = 0.0055

∴ C_{D_W} = 0.1900

Rocket: D = 0.75', L = 8'. H = 50 km = 164,000', T = 25°C

V = 3,000', a = 1160 ft/sec.

$\sigma = \frac{0.000837}{1.393}$ = 0.000621

V/a = 3.00

$\frac{\mu}{\mu_0} = \left(\frac{\ \ }{\ \ }\right)^{.76}$ = 1.0265.

$\frac{\rho}{\mu_e} = \frac{0.000621}{1.0265}$ = 0.000634

∴ R_e = 0.000634 × 8 × 3,000 × 6,900 = 1.0 × 10⁵

∴ $C_f = \frac{1.313 \times 10^{-3}}{\sqrt{11.0}}$ = 0.000320

C_{D_f} = 0.00360 × $\frac{32}{0.75}$ × 2.7 = 0.1230

C_{D_W} = 0.100.

Mean Free Path of Air Molecules

$$l = \frac{1}{\sqrt{2}\,\pi n s^2}$$

for air $s = 3.46 \times 10^{-8}$ cm.

$s^2 = 1.196 \times 10^{-15}$ cm.^{2}

$$\therefore\ l = \frac{1 \times 2.2414 \times 10^4 \times \frac{288}{273}}{\pi\sqrt{2} \times 1.196 \times 10^{-15} \times 6.064 \times 10^{23}}\ \frac{1}{\sigma}$$

where σ = density ratio.

$$\therefore\ l = 0.734 \times 10^{-5}\ \frac{1}{\sigma} =$$

If $H = 50$ km., $T = 15^\circ$, $\sigma = 0.000671$

$$l = \frac{0.734 \times 10^{-5}}{0.671 \times 10^{-3}} = 1.093 \times 10^{-2} \text{ cm.}$$

$$\delta\sqrt{\frac{U}{\nu_0 x}} = 9 \quad \text{for} \quad V/a = 3.00$$

$$\delta \cdot \frac{1}{x}\sqrt{\frac{Ux}{\nu_0}} = 9.$$

$$\frac{\delta}{x} = \frac{9}{\sqrt{\frac{Ux}{\nu_0}}} = \frac{9 \times 10^{-2}}{\sqrt{5.7}} = 3.78 \times 10^{-2} = 0.0378$$

1.14×10^{5}

x = say 5.7 at $4'$

$\delta = 0.0378 \times 4 \times 2.54 \times 12$

$= 4.63$ cm.

Fig. 1. No Legend

Fig. 2 Skin Friction Coefficients

(A) No Heat transfer to ~~the~~ Wall

(B) Wall Temperature ~~kept at one quarter to~~ ~~one~~ kept at 1/4 ~~to~~ that of free stream temperature

(C) Von Kármán's first approximation

Fig. 3. Velocity + Temperature ~~&~~ Distribution ~~there for the case of no~~ when no heat is transfered to ~~the~~ wall

Fig. 4. Velocity & Temperature Distribution ~~for the case of~~ when the wall temperature is kept at 1/4 ~~of that~~ of free stream temperature

Fig. 5 Heat Balance ~~for the case~~ when of the wall Temperature is kept at 1/4 of ~~the~~ of free stream temperature

Fig. 6. ~~Relative~~ Efficiency ~~High speed~~ ... of Cooling

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~~5) A. Busemann: loc. cit.~~

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~~9) For simplicity, the stars will be dropped in the following paragraphs.~~

~~10) H. Blasius: loc. cit.~~

8 ~~10~~) R.H. Kent: The Rôle of Model Experiment in Projectile Design. Mechanical Engineering, Vol. 54, p. 641 - 646, 1932.

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5) This problem was also discussed by L. Crocco in following paper
Su di un valore massimo del coefficiente di trasmissione del calore da una lamina piana a un fluido scorrente. Rendiconti R. Accademia dei Lincei Vol. 14. fasc. 490 – 496, 1931

(See reverse side of page)

Boundary Layer in Compressible Fluids

1

Th. von Kármán and H.S. Tsien

The introduction of variable density makes the solution of flow problems in general very difficult; hence every case in which an exact or even an approximative solution of the equations of the motion of compressible fluid can be obtained has considerable theoretical interest. Several authors noticed that the theory of the laminar boundary layer can be extended to the case of compressible fluids moving with arbitrarily high velocities without insurmountable mathematical difficulties being involved. Bustmann[1] the equations and calculated a singular velocity profile for one speed ratio. (Under speed ratio we understand the ratio of the air speed to the velocity of sound). Frankl's[2] paper is very complicated and uses sometimes arbitrary methods of approximation. The senior author[3] first obtained a first approximation by simple calculation, but apparently not sufficiently exact. Hence in the first part of the present paper a better method for the solution of the problem is developed.

The boundary layer theory for very high velocities is not without practical interest. First the statement can be often found in technical and semi-technical literature

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on rockets and similar high speed devices that in the case of high speed the skin friction becomes more and more insignificant at high speeds. Of course it is known that with increasing Reynolds number the skin friction coefficient is decreasing i.e. the skin friction becomes relatively small in comparison with drag produced by wave formation or direct shock. However one has to be into account that high speed flight will be mostly performed at high altitude, i.e. in air of very low density and therefore the Reynolds number is necessarily relatively small or has moderate values in spite of high speed.

Another interesting point in the boundary layer of compressible fluids is the thermodynamic aspect of the problem. In the case of ordinary speeds the influence of the heat produced in the boundary layer can be neglected for calculation of the drag and of the heat transfer. In the case of very high speeds the heat produced in the boundary layer is not negligible in comparison with the heat transfer between say a hot stream and a cold wall, but as a matter of fact it determines the development of heat flow.

In the second part of the paper a few simple examples of heat flow through the boundary layer are discussed.

The assumption of laminar flow on most part of the paper might be excused by the remarkable state of our knowledge concerning laws of turbulent flow of compressible fluids at high speeds. Fuller excuse is given by the fact that — as mentioned above — in many problems of the type treated in the paper where the results of the paper can be applied, Reynolds number is relatively small such that a considerable portion of the boundary layer is probably, de facto, laminar.

Ackeret[1] called the attention to the possibility that the stability conditions in supersonic flow might be quite different from those we observe in flow with moderate velocities. The authors also believe that the stability considerations as developed by Tollmien and others cannot be applied without modification to the case, in which very high velocities are involved and turbulence might be delayed by the compressibility effect.

Finally, some conclusions of the paper, as it will be pointed out, are also applicable to turbulent flow and in other cases like in the calculation of drag, the laminar friction gives surely at least a correct lower limit.

~~Laminar~~ Boundary Layer ~~of~~ in Compressible Fluids

Th. von Kármán and H.S. Tsien

The introduction of ~~varying~~ variable density ~~to the study of flow of fluids~~ makes the ~~general solution~~ solution of flow problems in general very difficult. ~~even without considering viscosity~~ However, for the case of laminar boundary layer, however, the solution of the equations of motion of compressible fluids seems fairly simple + ~~furthermore, As the high speed flight for which the effect of compressibility is ~~mostly felt~~ appreciable will probably conducted in future at very high altitude, ~~where~~ due to greatly reduced density, ~~the~~ Reynolds number ~~will~~ be low, ~~then~~ the problem of laminar skin friction ~~will probably~~ is not ~~quite~~ without practical interest predominate the total skin friction.~~

Busemann[1], Fränkl[2] and von Kármán[3] have made

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1) A. Busemann: Gasströmung mit laminarer Grenzschicht entlang einer Platte. Z.A.M.M. Vol. XV, p.23, 1935

2) Frankl: Laminar boundary layer of compressible fluids Trans. of the Joukowski Cent. Aero-Hydrodynamical Inst. Moscow, 1934 (Russian)

3) von Kármán: The Problem of Resistance in Compressible Fluid, in Convegno di Scienze, Fisiche, Mathematiche e Naturali (Tema: Le alte velocità in aviazione) Reale accademia d'Italia, Rome.

some investigations in this problem. The paper uses rather complicated computations. The following work is an attempt to study this problem more thoroughly, and to check the previous calculations.

(1)

If the x-axis is taken along the plate in the direction of outside uniform flow, and y-axis perpendicular to the plate; also u, v indicate the x & y components of velocity at any point, the simplified equation of motion in the boundary layer is

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \qquad (1)$$

where ρ is the density, μ the viscosity; both are supposed (to be) variable.

The equation of continuity in this case is

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \qquad (2)$$

A third equation determines the energy balance between heat produced due to viscous dissipation and heat transferred by conduction and convection. With same simplification as used in equations (1) + (2), we can write

$$\rho u \frac{\partial}{\partial x}(c_p \cdot T) + \rho v \frac{\partial}{\partial y}(c_p \cdot T) = \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \qquad (3)$$

where c_p is the specific heat at constant pressure & λ is the coefficient of heat conduction. If we assume Prandtl's number, $\frac{c_p \mu}{\lambda}$, is equal to 1, then it can be easily shown that both (1) & (3) can be satisfied by putting the temperature T to be a certain parabolic function of velocity u only. This relation between T & u is

$$\frac{T}{T_0} = \frac{T_w}{T_0} - \left(\frac{T_w}{T_0} - 1\right)\frac{u}{U} + \frac{k-1}{2}M^2 \frac{u}{U}\left(1 - \frac{u}{U}\right) \qquad (4)$$

where T_w = temperature at the wall of the plate.
T_0 = Temperature of outside uniform flow.
M = Mach's number of outside flow.

We have then

$$\frac{1}{T_0}\left(\frac{\partial T}{\partial y}\right)_w = \frac{1}{U}\left[\frac{k-1}{2}M^2 - \left(\frac{T_w}{T_0} - 1\right)\right]\left(\frac{\partial u}{\partial y}\right)_w \qquad (5)$$

where the subscript "w" refers to conditions at the surface of the plate. Now $\left(\frac{\partial u}{\partial y}\right)_w$ is always positive, therefore if $\frac{k-1}{2}M^2 > \frac{T_w}{T_0} - 1$, heat is transfered from the fluid to the wall; if $\frac{k-1}{2}M^2 = \frac{T_w}{T_0} - 1$, there is no heat transfer between fluid & wall and if $\frac{k-1}{2}M^2 < \frac{T_w}{T_0} - 1$, heat is transfered

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from the plate to the fluid. For the second case, the energy content per unit mass $\frac{u^2}{2}+C_pT$ is constant ~~over the whole~~ for the whole region of the boundary layer.

The relation between ρ and T is

$$\rho = \rho_0 \frac{T_0}{T} \quad \text{based on the kinetic theory of gases} \tag{6}$$

For viscosity, ~~theoretical~~ considerations leads to

$$\mu = \mu_0 \left(\frac{T}{T_0}\right)^{\frac{1}{2}} \tag{7}$$

But the following formula fits the actual data better

$$\mu = \mu_0 \left(\frac{T}{T_0}\right)^{0.76} \tag{7a}$$

Busemann[1] taking the limiting case where $\frac{k-1}{2}M^2 = \frac{T_w}{T_0} - 1$, & using (7), found that for high Mach's number, the velocity profile is approximately linear with the exception of the outside boundary of the layer where u is nearly equal to U. von Kármán[2] used this linear velocity profile and the integral relation between friction & momentum; he found that by (7),

$$C_f = \frac{\text{Frictional Force per unit width of plate}}{\frac{\rho}{2}U^2 \text{ length of plate}}$$

$$= \sqrt{8f(\lambda)}\sqrt{\frac{\mu_0}{\rho U x}}\left[1+\frac{k-1}{2}M^2\right]^{-\frac{1}{4}} \tag{8}$$

1) loc. cit

2) " "

If (7a) is used, then

$$C_f = \sqrt{8f(\lambda)}\sqrt{\frac{\mu_0}{\rho U x}}\left[1+\frac{k-1}{2}M^2\right]^{-0.12} \qquad (9a)$$

In both expression $\sqrt{8f(\lambda)}$ is found to have the following values.

Table 1.

M	0	1	2	5	10	∞
$\sqrt{8f(\lambda)}$	1.16	1.20	1.25	1.39	1.50	1.57

It is evident that this approximation is not satisfactory for small values of Mach's number. For $M=0$, the case is same as "Blasius'"[1] solution of incompressible fluid, and $\sqrt{8f(\lambda)}$ should be 1.328. This result is included in Fig. 3.



To solve the problem more rigorously, we have to resort to (1) + (2). By introducing the stream function ψ defined as

$$\frac{\rho}{\rho_0}u = \frac{\partial\psi}{\partial y}, \qquad -\frac{\rho}{\rho_0}v = \frac{\partial\psi}{\partial x},$$

1) H. Blasius: Grenzschichten in Flüssigkeiten mit kleiner Reibung. Zeit. f. Math. u. Phys. Bd. 56, p. 1., 1908

the equation of continuity is satisfied. Then ψ is taken as independent variable as von Mises[1] used in his simplification of boundary layer equation of incompressible fluid, we have

$$\frac{\partial u^*}{\partial x^*} = \frac{\partial}{\partial \psi^*}\left[u^* \rho^* \mu^* \frac{\partial u^*}{\partial \psi^*}\right] \quad [2] \tag{9}$$

where

$$\left.\begin{aligned} u^* &= \frac{u}{U}, \\ x^* &= \frac{x}{L}, \\ \psi^* &= \frac{\psi}{UL}\sqrt{\frac{\rho_0 U L}{\mu_0}} \\ \rho^* &= \frac{\rho}{\rho_0} \\ \mu^* &= \frac{\mu}{\mu_0} \end{aligned}\right\} \tag{9a}$$

and L is a convenient length, say length of the plate.

(9) can be further simplified by introducing a new independent variable $\zeta = \frac{\psi}{\sqrt{x}}$, then

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1) von Mises: Bemerkung zur Hydrodynamik ZAMM Vol. VII, p. 425, 1927

2) For simplicity, the stars will be dropped in following paragraphs.

$$-\frac{\zeta}{2}\frac{du}{d\zeta} = \frac{d}{d\zeta}\left[u\rho\mu\frac{du}{d\zeta}\right] \qquad (10)$$

This can be solved by method of successive approximation. We can start with the known Blasius[1] solution. Then ρ & μ, being a function of temperature only, are in turn functions of u by (6), (7) or (7a), the whole right-hand side of (10) can be expressed in terms of ζ. Therefore, we can put

$$u\rho\mu = f(\zeta).$$

Consequently

$$u = C\int_0^{\zeta}\frac{e^{-\int_0^{\zeta}\frac{\zeta d\zeta}{f}}}{f}\,d\zeta \qquad (11)$$

where C is determined by the boundary condition,

$$\frac{1}{C} = \int_0^{\infty}\frac{e^{-\int_0^{\zeta}\frac{\zeta d\zeta}{f}}}{f}\,d\zeta \qquad (11a)$$

From (11) we can make a second approximation. For the cases investigated, three calculations will usually give results close to three digits, & even the worst case, four calculations are enough.

Having computed the final u^*, the y corresponding to u can be calculated by[1]

$$y\sqrt{\frac{U\rho_0}{\mu_0 x}} = \int_0^{\zeta}\frac{d\zeta}{\rho u} \qquad (12)$$

---

1) Loc. cit.

7a

This can be solved by the method of successive approximation, as δ^* and μ^* are functions of temperature only as shown in Eqs. (6), and (7) or (7a) and the temperature is a function of u^* then by starting with the known Blasius solution (Ref 5) the right-hand side of Eq (10) can be expressed in terms of ξ. Therefore one can write

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Also the skin friction can be computed by the momentum law,

$$C_f = \frac{F}{l_0 \frac{\rho_0}{2} U^2} = \frac{2\int_0^\infty (1-u)\,d\xi}{\sqrt{R}} \tag{13}$$

For the case $\frac{k-1}{2}M^2 = \frac{T_w}{T_0} - 1$, using (the approximate viscosity relation) (7a), the velocity profile, temperature distribution & frictional drag coefficient are calculated for different values of Mach's number of outside flow. The results are shown in Fig. 2, Fig 3 and Fig. 4. The velocity profiles for high speeds is very nearly a straight line, but it can be seen that the wall temperature for greater Mach's number is very high. If the free stream temperature is 40°F., then the wall temperature will be 1,600°F, 3,620°F, 6590°F or 10,170°F for Mach number of 4, 6, 8 or 10. Therefore it is without doubt that the law of viscosity as shown in (7a) will not hold. Also under such high temperature, the heat transfer due to radiation can not be neglected. Therefore the results shown here for extreme Mach's numbers are qualitative only.

The change in the constant $C_f\sqrt{R}$ is appreciable but not great, a decrease from 1.328 for M=0 to 0.925 for M=10, or about 30%. The variation however for the region of M from 0 to 3 is much smaller. ~~However,~~ in any practical application, the frictional drag for supersonic flow will be overshadowed by the large wave drag of thick wave resistance.

Fig. 4 also shows that the linear distribution (4a) approximates fairly the ~~ss~~ more exact calculation ~~rather closely~~, ~~except~~ for very high Mach number.

9a

(IV)

In order to examine the conditions prevailing when a hot gas flows over a cooled plate, another series of calculation is made with the condition that $T_w = 0.25\,T_0$. ~~Again using that we have~~ the results shown in Fig. 5, Fig. 6. ~~& Fig. 7.~~ ~~These~~ curves show ~~the effect of high speed better than~~ Part (III), since the wall ~~temperature~~ is kept at a constant ratio of ~~outside~~ free stream temperature.

9a

As an example let us consider the case of a shell and the case of a ~~[illegible]~~ rocket. Taking the diameter of shell to be 6 inches total length 28 inches and the velocity of shell to be 1500 ft/sec, the altitude = 32,800 ft (10 km), the Reynolds number based on the length of shell will be 7.46×10^6 (which is [illegible] and the speed ratio [illegible]) is 1.52. Then from fig 6, $C_f = 1.282 \times 10^{-3} / \sqrt{7.86}$ = 0.000459. Changing the skin friction coefficient (based on the skin area) to drag coefficient (based on the maximum cross section), we have

$$C_{D_f} = 0.0355$$

The drag coefficient due to wave formation taken from Kent's[1] experiment is

$$C_{D_w} = 0.190$$

Therefore the ratio of skin friction to wave resistance is $\frac{0.0355}{0.190} = 0.179$.

However, the condition is completely changed in the case of rocket. Let us assume a diameter of 9" & length

1) R. H. Kent: The Rôle of Model Experiment in Projectile Design. Mechanical Engineering. Vol. 54. p. 641-646 (1932)

of ℓ feet, also assume the altitude of flight be 50 km (164,000 feet), and velocity be 3400 ft/sec. ∴ the density ratio at that altitude is 0.00017 & temperature is 25°C. (deduced from data in meteor) then the Reynolds number is 2.14×10^{5} (which is within the range of laminar boundary layer of incompressible fluid), and speed ratio is 3.00. From Fig 4, the skin friction coefficient is

$$C_f = 1.213 \times 10^{-2}/\sqrt{11.4} = 0.00360$$

Changing into drag coefficient, we have

$$C_{D_f} = 0.123.$$

The drag coefficient due to wave formation from Kent's experiment at this speed ratio is

$$C_{D_w} = 0.100.$$

Therefore ratio of skin friction & wave resistance is now $0.123/0.100 = 1.23$. This shows clearly the importance of laminar skin friction in case of a slender body moving with high speed in extremely rarefied air. It also proves the fallacy

of belief that wave resistance would be the predominating part in the total drag of a body moving with velocity higher than that of sound. The reason underlying this fact can be easily understood when one recalls that the wave resistance of a body is approximately directly proportional to the velocity while the skin friction is proportional to 1.5th power to square of velocity. Hence for very high velocity, the wave resistance [illegible] be only a negligible portion of the total drag of the body. (combined with high kinematic viscosity)

therefore the ratio of skin friction & wave resistance is increases with the speed. With very high velocity & high kinematic viscosity, the wave resistance may even be a negligible portion of the total drag of the body.

II.

In order to give an idea of the thermodynamical aspect of the problem the flow of a hot gas along a cold surface will be considered. In the first part [of this] paper it was assumed that there is no heat flow through the surface of the wall and it was found that [illegible] the wall temperature rises up to the temperature which corresponds to the total energy content (kinetic energy + heat content) of the gas in the free stream. We now consider flow near the wall cooled from outside at such extent that the wall temperature is kept constant. The actual calculations have been carried out for the assumption $T_w = \frac{1}{4} T_0$, i.e. that the absolute temperature of the wall is one fourth of the absolute temperature of the gas, for instance $T_0 = 2000°$, $T_w = 500°$ ($t_0 = 1727$, $t_w = 225°C$). With the same assumption (on the variation of μ) as in Part I, we obtain the results as shown in Fig 5, & Fig 6. The drag curve is plotted in Fig 4, in comparison with the case of insulated wall.

2000
273
1727

(10)

Also in this case the highest temperature in the boundary layer is very high for extreme Mach number. The character of variation of $C_f\sqrt{R}$ with M is similar to that obtained in the case without heat conduction through the wall, shown in (II)

(10a) ~~By consideration of~~ The initial slope of the velocity profile ~~we have~~ is equal to

$$\left(\frac{\partial u}{\partial y}\right)_w = \frac{U}{L}\frac{\sqrt{R}}{\sqrt{\pi^*}}\left(\frac{\mu_0}{\mu_w}\right)\frac{\sqrt{R}\,C_f}{4} \qquad (14)$$

~~By differentiation of equation (4) the relation between~~ velocity slope and temperature gradient can be obtained and using (14) the latter will be equal to

$$\left(\frac{\partial T}{\partial y}\right)_w = K\,\frac{T_0\sqrt{R}}{4L\sqrt{\pi^*}} \qquad (15)$$

where $K = \frac{\gamma}{2}^{0.76}\left(0.75 + \frac{k-1}{2}M^2\right)\sqrt{R}\,C_f$

Therefore the heat conducted ~~away from the wall~~ transferred to ~~the wall~~ for a strip of width ~~& length~~ of the wall of length L is equal to

$$Q = \int_0^L \left(\lambda\frac{\partial T}{\partial y}\right)_w dx = \frac{K\lambda_w T_0\sqrt{R}}{2L}\int_0^L \frac{dx}{\sqrt{\pi^*}}$$

or approximately

$$Q \cong K\lambda_w T_0\sqrt{R} \qquad (16)$$

The value of K is given in Table II.

Table II

M	K
0	1.53
1	1.93
2	3.12
5	10.53
10	33.98

However the temperature maximum occurs at a certain distance from the wall inside of the boundary layer

10a

~~However, the [illegible] temperature maximum is at a certain distance from the wall and therefore [illegible] the heat flow can be determined as follows. Certain amount of heat is always~~

The heat transferred from the boundary layer to the wall can be calculated as follows.

× ——— page 10.

An interesting general relation between the heat transferred through the wall surface and the frictional drag can be obtained using the assumption, that Prandtl's number — i.e. the ratio $\frac{\lambda g}{\mu c_p}$ — is equal to unity. The same assumption was used also in the previous calculation. It is remarkable that the relation holds also as well for laminar as for turbulent flow.

The heat flow q per unit time and unit area of the wall surface is $q = \lambda_w \frac{\partial T}{\partial y}$, and the frictional drag (τ) per unit area is $\tau = \mu_w \frac{\partial u}{\partial y}$. Using Eq. (4) the ratio $\frac{q}{\tau}$ can be calculated from the relation (Eq. 5)

$$\frac{q}{\tau} = \frac{\lambda_w}{\mu_w} \frac{T_0}{U} \left[\left(1 - \frac{T_w}{T_0}\right) + \frac{\kappa - 1}{2} M^2 \right] \qquad (17)$$

2

~~Hence~~

In the case $T_0 > T_w$, i.e. when the wall is colder than the free stream ~~if a hot fluid is cooled by a colder surface~~, the effect of compressibility is to increase the heat transferred through the wall. However it would be erroneous to ~~conclude that~~ interpret this result as an "improvement" of cooling, ~~...~~, because at high speeds the heat produced in the boundary layer is of the same order as the heat transferred through the wall, ~~and~~ in order to determine the efficiency of the cooling a complete heat balance must be made ~~is to be considered~~. For this purpose equation (17) ~~is not~~ does not give sufficient information and the velocity and the temperature distribution in the boundary layer must be computed. Such calculations were carried out ~~for the particular assumption~~ (that $T_w = \frac{T_0}{4}$ ~~i.e. for the particular case that the absolute temperature of the fluid is four times the absolute temperature of the wall~~ and are ~~The result is shown in Fig 5 The ordinate gives heat for the dimensionless heat per unit time and unit length. The~~

The complete heat balance is ~~made~~ shown in Fig. 5

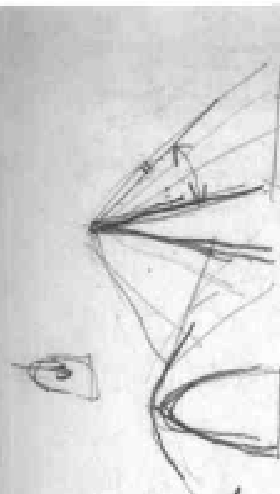
"dissipation" curve represents ~~the loss~~ in dimensionless form the heat ~~to~~ produced by friction per unit time and unit width of the plate, the lower curve shows the increase (or decrease) of the heat content ~~...~~ per unit time and unit width. ~~R denotes Reynolds' number~~ It is seen that cooling takes place for $M < 2.6$. Beyond this limit more heat is produced by friction than the amount which can be transferred to the ~~(cold)~~ wall and ~~the fluid is~~ as a matter of fact the fluid is heated.

The difference of the ordinates corresponds to the heat transferred through the wall.

3

In the case $T_w > T_0$ i.e. when a hot wall is to be cooled
by a colder fluid, the heat transfer decreases with increasing
Mach's number. This is graphically represented in diagram
Fig 6. , where the ordinate represents the
ratio between (heat transfer) the factor the actual
drag heat transfer and the value calculated
without compressibility effect.
The calculation is carried out for a fluid
temperature of −55°F for the cases wall temperature
equal to 180°F and 303°F. It is seen that cooling becomes
zero for M = 1.69 in the former and for M = 2.08 in the latter case.
However the decrease of cooling efficiency is appreciable
even at much lower speeds. This emphasizes the benefit of
the reduction of the speed of cooling air and the relatively
poor efficiency of cooling surfaces exposed directly to high
speed airstream. The diagrams
apply to laminar as well as to turbulent motion.

(The curves in Fig 6)
being derived from equation (17)



Boundary Layers in Compressible Fluids

By

Th. von Kármán and H. S. Tsien

Summary

~~The significance of~~

First part of this paper is concerned with the laminar boundary layer in compressible fluids. Its significance ~~is first pointed out~~ in practical applications ~~in the first part of this paper~~ is first pointed out in the introduction. Then the general character of the ~~to lamina~~ boundary layer along a flat plate is discussed, ~~with the help~~ with ~~special notice to~~ especially ~~regard to~~ the thermodynamical aspect of the problem. The first approximation to the ~~solution~~ calculation of skin friction ~~of equations of motion~~ as ~~developed~~ developed by the senior author is then briefly explained ~~summarized~~. ~~Then a more rigorous solution is~~ Due to the unsatisfactory result of this method for low Mach number a more rigorous solution using method of successive approximation is ~~explained~~ developed, and the results of calculations for the case of an insulated plate in a high speed flow are shown. ~~The results calculated show~~ The effect of compressibility on skin friction is appreciable but not very large, however the velocity ~~distribution~~ &

(back page)

The calculated skin friction coefficients are then used to find the drag of a projectile and a wingless sounding rocket. The importance of skin friction in high speed flight at very rarefied air is thus clearly brought out.

In the second part of this paper, the problem of heat transfer to or from the wall of a plate is studied in detail. Results of calculations for the case of flat plate cooled to a temperature $\frac{1}{4}$ of free stream are shown. The velocity & temperature distributions differ appreciably from the case of insulated plate, & skin friction is higher. The heat transfer increases rapidly with speed. However, due to fact that the simultaneous increase in viscous dissipation is even higher, so for high Mach number a large amount of heat remains in the gas, & the gas is heated instead of being cooled. A similar situation arises in the case of a radiator. At high Mach's number, the boundary layer is heated up to such an extent that the radiator is no longer being cooled. An numerical example with condition similar to

(back page)

A heat balance is calculated for the case of wall plate temperature $\frac{1}{4}$ of free stream

Plan of Study of Boundary Layer in Compr. Fluid. ①

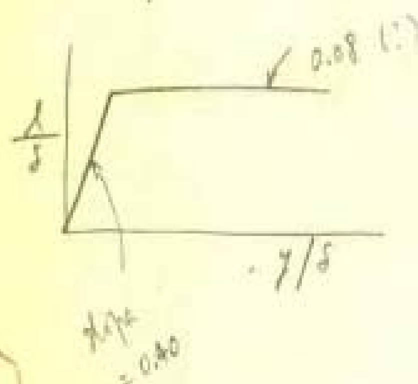
(I) Laminar Boundary Layer over flat plate

Solution of the Boundary Layer equation by numerical integration. For all cases of $(T_w - T_0)$

(II) Calculation of Critical Reynolds No.

(III) Solve the Turbulent boundary layer equation by using the equation

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\rho\left(l \frac{\partial u}{\partial y}\right)^2\right]$$



(II_0) Study of turbulent boundary layer in pipe.

②

with Pressure gradient

$$\begin{cases} \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left(\mu \cdot \frac{\partial u}{\partial y}\right). \\ \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \end{cases}$$

The energy equation

$$\rho u \cdot \frac{\partial}{\partial x}(c_p T) + \rho v \cdot \frac{\partial}{\partial y}(c_p T) - u \frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) + \mu \cdot \left(\frac{\partial u}{\partial y}\right)^2.$$

---

check the left hand side —

$$c_p = c_v + R = c_v + \frac{p}{\rho T} \quad . \quad c_p T = c_v T + \frac{p}{\rho}.$$

$$\rho u \cdot \frac{\partial}{\partial x}(c_p T) = \rho u \cdot \frac{\partial}{\partial x}\left(c_v T + \frac{p}{\rho}\right) = \rho c_v u \frac{\partial T}{\partial x} + u \frac{\partial p}{\partial x} - \frac{p u}{\rho} \frac{\partial \rho}{\partial x}$$

$$\rho u \cdot \frac{\partial}{\partial y}(c_p T) = \rho c_v v \frac{\partial T}{\partial y} + v \frac{\partial p}{\partial y} - \frac{p v}{\rho} \frac{\partial \rho}{\partial y}$$

$$\therefore \quad \rho u \cdot \frac{\partial}{\partial x}(c_p T) + \rho u \cdot \frac{\partial}{\partial y}(c_p T) - u \frac{\partial p}{\partial x}$$

$$= \rho c_v \left[u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}\right] - \frac{p}{\rho}\left[u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y}\right].$$

due to $\frac{\partial p}{\partial y} \to 0$. Also by continuity,

therefore $-\left[u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y}\right] = + \rho\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]$, so

the left side of energy equation

$$= \rho c_v \left[u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}\right] + p\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] \qquad \text{O.K.}$$

Assume $\lambda = c_p \cdot \mu$, and $c_p T = f(u)$, we have

$$\left(\rho u \cdot \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y}\right) \cdot f'(u) - u \frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left(\mu \cdot \frac{\partial u}{\partial y}\right) \cdot f'(u)$$

$$+ \mu \cdot \left[f''(u) + 1\right] \cdot \left(\frac{\partial u}{\partial y}\right)^2.$$

③

By making $f''(u) = -1$, and $f'(u) = -u$, we have

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left(\mu \cdot \frac{\partial u}{\partial y}\right)$$

Therefore the energy equation will be satisfied when $f'(u) = -u$

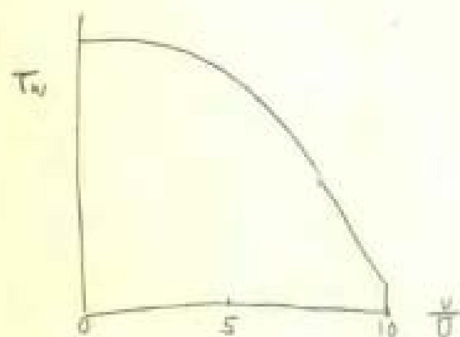
$$f(u) = c_p T = -\frac{1}{2}u^2 + A$$

When $u = 0$, $T = T_w$, so $c_p T_w = A$, or

$$\therefore \quad T = -\frac{1}{2c_p}u^2 + T_w = T_w - \frac{u^2}{2c_p} = T_0 + \frac{1}{2}\Big(\frac{U^2}{c_p}$$

When $u = U$, $T_0 = -\frac{1}{2c_p}U^2 + T_w$

$$\therefore \quad (T_w - T_0) = \frac{U^2}{2c_p}$$



④

With Pressure Gradient.

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)$$

By $\frac{\partial}{\partial y} = \rho u \frac{\partial}{\partial \psi}$, $\frac{\partial}{\partial x} = \frac{\partial}{\partial n} - \frac{v}{u}\frac{\partial}{\partial y}$, we have, since $\frac{\partial p}{\partial y} \to 0$,

$$\rho u \frac{\partial u}{\partial n} + \frac{\partial p}{\partial n} = \rho u \frac{\partial}{\partial \psi}\left[\mu \rho u \frac{\partial u}{\partial \psi}\right].$$

At outside flow, Bernoulli's equation is satisfied, so

$$U \frac{\partial U}{\partial n} = -\frac{1}{\rho_0}\frac{\partial p}{\partial n}.$$

where $\rho_0 = \rho$ at outer edge of boundary layer.

or $\frac{\partial p}{\partial n} = -\frac{\rho_0}{2}\frac{\partial U^2}{\partial n}$. so

$$\frac{\rho}{2}\frac{\partial u^2}{\partial n} - \frac{\rho_0}{2}\frac{\partial U^2}{\partial n} = \rho u \frac{\partial}{\partial \psi}\left[\mu \rho \frac{\partial u^2}{\partial \psi}\right]$$

$$\frac{\partial}{\partial n}\left(\frac{\rho}{2}u^2 - \frac{\rho_0}{2}U^2\right) - \frac{u^2}{2}\frac{\partial \rho}{\partial n} = \rho u \frac{\partial}{\partial \psi}\left[\mu\left\{\frac{\partial}{\partial \psi}\left[\frac{\rho}{2}u^2 - \frac{\rho_0}{2}U^2\right] - \frac{u^2}{2}\frac{\partial \rho}{\partial \psi}\right\}\right]$$

Let $z = \frac{\rho}{2}u^2 - \frac{\rho_0}{2}U^2$, we have, letting $\frac{\partial}{\partial n} = \rho_0 U \frac{\partial}{\partial \varphi}$,

$$\rho_0 U \frac{\partial z}{\partial \varphi} - \frac{u^2}{2}\rho_0 U \frac{\partial \rho}{\partial \varphi} = \rho u \frac{\partial}{\partial \psi}\left[\mu\left\{\frac{\partial z}{\partial \psi} - \frac{u^2}{2}\frac{\partial \rho}{\partial \psi}\right\}\right]$$

or

$$\frac{\partial z}{\partial \varphi} - \frac{u^2}{2}\frac{\partial \rho}{\partial \varphi} = \frac{\rho u}{\rho_0 U}\frac{\partial}{\partial \psi}\left[\mu\left\{\frac{\partial z}{\partial \psi} - \frac{u^2}{2}\frac{\partial \rho}{\partial \psi}\right\}\right]$$

1.1.2
Kármán—钱近似

1.1.2.1
Two – Dimensional Subsonic Flow of Compressible Fluids
可压缩流体的二维亚声速流动

这是发表于 1939 年的"Two – Dimensional Subsonic Flow of Compressible Fluids"（可压缩流体的二维亚声速流动）一文的部分手稿，共有 17 页。这一工作是作者博士论文的一部分。

在提高飞机飞行速度的努力中，在机翼的空气动力学设计中，计算翼面的压力分布遇到了困难。对于平面超声速流动，可以采用已有的特征线法计算机翼上的压力分布；可是当飞行速度低于声速的时候，已有的方法只能计算机翼很薄或速度较低的情况。

1902 年，俄国的 S. A. Chaplygin（查普雷金）在他的博士论文中对定常位势流动方程作了一个变换，将自变量从物理平面 (x, y) 变换到速度平面 (q, θ)，称为速度图法，把原来的非线性的方程化为线性方程。作为近似，他又建议将等熵关系曲线用它的切线来代替，进一步简化了方程。后来，Demtchenko（丹姆千科）（1932）和 Busemann（布兹曼）（1933）用了驻点处的切线作了近似计算，可惜只适用于飞行速度小于 0.5 倍声速的情况。

Theodore von Kármán（冯·卡门）凭着对物理问题的洞察力，建议作者在求解由速度图变换得到的线性方程时，用来流状态处的切线作近似，结果可能更好。钱学森证明，虽然同样是切线近似，采用 Kármán— 钱所用的来流状态处的切线近似，可以计算高亚声速的流动，而且得到很精确的计算结果。原因在于：在流场的大部分区域，

流速和声速的数值更接近于来流的数值，而不是接近于驻点处的数值。在第二次世界大战期间以及战后一个相当长的时期，在现代计算手段——电子计算机出现以前，作者这一近似计算方法被广泛应用于飞机翼型的设计，且被人们称为“Kármán—钱近似”方法。

①

General Discussion on the Flow of Compressible Fluids

By Prandtl

§1 Preliminary Considerations

The problem of fluid motion is already very complicated even with the assumption that it is incompressible. So with the introduction of compressibility, we must obtain a simplification in the other direction, i.e., we assume that the fluid is nonviscous. Therefore we generally neglect the viscosity and will treat the nonviscous compressible fluid. Furthermore, we assume that the density only dependents on the pressure. The case of inhomogenities arised from such factors as heat introduced by conduction and heat developed by combustion in the fluid, must be excluded. Therefore we assume that the relation between the pressure p and the density ρ is definite & single-valued.

In such a nonviscous homogeneous compressible fluid, the Lagrange's theorem, i.e., the fluid without rotation is always without rotation, also hold as in the nonviscous homogeneous incompressible fluid. However this theorem only holds when the flow is continuous. In flow with supersonic velocity, there is discontinuous motion with irreversible process of definite raise in entropy. Here the homogeneity is disturbed, & the

behind this discontinuity cannot obey the Lagrange's theorem.

Since the rest condition is a special case of irrotational motion, so if we exclude the cases mentioned before, the motion produced from a rest state, either steady or unsteady must be also irrotational. Irrotational motion can always be expressed through velocity potential, as the velocity can be expressed as gradient of the potential, as

(1) $\mathfrak{w} = \operatorname{grad} \phi$.

In its components, we have

(1a) $u = \frac{\partial \phi}{\partial x}$, $v = \frac{\partial \phi}{\partial y}$, $w = \frac{\partial \phi}{\partial z}$.

For such potential motion of homogeneous nonviscous fluid, the Bernoulli's equation holds,

(2) $\frac{\partial \phi}{\partial t} + \frac{\mathfrak{w}^2}{2} + P - U = f(t)$,

where $P = \int \frac{dp}{\rho}$, the pressure function

U = the force function, in case of gravitational force $U = -gz$;

$f(t)$ = any function of time.

The second equation is the equation of continuity, which expresses the constancy of mass. We can express it either for a fluid element bounded by the elementary volume, the mass of this fluid element is constant, as

③

for a fixed volume element in the space, there is as much fluid mass flows in per unit time as there is fluid mass flows out per unit time. Either consideration leads to the equation

(3). $$\frac{\partial \rho}{\partial t} + div(\rho \overline{W}) = 0$$

We can also put $div(\rho\overline{W}) = \rho\, div\, \overline{W} + \overline{W}\cdot grad\, \rho$.

Equation (2) & (3) together define our problems.

If we exclude the cases of applications to meteology, then the role played by gravity is always of second order, so the term U in the equation is always neglected. In the cases treated in acoustic, for the small but rapidly changing velocity the squared power term $\left[\frac{\overline{W}^2}{2}\right.$ in eqn. (2) & $\overline{W}\cdot grad\,\rho$ in Eqn. (3)$\left.\right]$ can be neglected.

By consideration of Eq. (1), & since $div.\, grad\, \phi = \Delta\phi$, we have from Eq. (3)

(3a) $$\frac{\partial \rho}{\partial t} + \rho\Delta\phi = 0$$

We have then differentiate Eq. (2) with respect to the time.

We have (2a) $$\frac{\partial P}{\partial t} = \frac{1}{\rho}\cdot\frac{dp}{d\rho}\cdot\frac{\partial \rho}{\partial t} = \frac{c^2}{\rho}\frac{\partial \rho}{\partial t}.$$

$\frac{dp}{d\rho}$ has the dimension of square of velocity, & so we put $\frac{dp}{d\rho} = c^2$, where c is a function of ρ. $f(t)$ is constant when we consider the fluid is at rest at infinity. Therefore the differentiation of Eq (2) gives

⑨

(4) $$\frac{\partial^2 \Phi}{\partial t^2} + \frac{c^2}{\rho}\frac{\partial \rho}{\partial t} = 0.$$

The combination of (3a) & (4) give

(5) $$\frac{\partial^2 \Phi}{\partial t^2} = c^2 \Delta \Phi,$$

the well known differential equation of acoustic. One well known solution of this equation is the plane sound wave, which can be expressed as

(6) $$\Phi = F(x - ct).$$

The function F is perfectly arbitrary, but all the waves propagate by a velocity c. Another solution is the spherical wave, which can be expressed as

(7) $$\phi = \frac{1}{r} f(r - ct),$$

where r means the distance from the center. The potential for a cylindrical wave is somewhat less complicated. It can be written as

(8) $$\Phi = \int_{\xi_1}^{\xi_2} \frac{f(\xi)\,d\xi}{\sqrt{(ct-\xi)^2 - r^2}}.$$

The meaning of the velocity c as the velocity of propagation is evident from these solutions. We call this velocity, which from above consideration is also a function of density, the velocity of sound. The reason that we take this velocity c as a constant in the above

(5)

treatment is due to the fact that with the assumption that the amplitude of this oscillation is small, & so the deviation of ρ from the initial value ρ_0 is negligible. For any finite amplitude, we must go back to Eqs. (2) & (3) for a rigorous solution. This was done by Riemann (1), & shows that the compression propagate faster than the rarefication, and therefore the wave front grows steeper & steeper till a continuity is produced. This we call "compression shock" or "shock wave". The compression flows forward, while the rarefication flows backwards, and the shock itself propagates with a velocity not very different from the sound velocity. Therefore the relative velocity of the shock & the compression is smaller than sound velocity while the relative velocity of the shock & the rarefication is greater than sound velocity.

2. Steady Potential Flow

⑥

For the unsteady sound motion of small free motion, the above qualitative treatment will be sufficient. Now we will go to the chief subject of this treatment, i.e. the steady potential flow. When we again neglect the gravity, we have the equation (2) + (3) changed to the form

$$(2b) \qquad \frac{\mathfrak{W}^2}{2} + P(\rho) = \text{constant}$$

and

$$(3b) \qquad \operatorname{div} \mathfrak{W} + \frac{1}{\rho}\mathfrak{W}\cdot \operatorname{grad}\rho = 0.$$

But $\frac{1}{\rho}\operatorname{grad}\rho = \frac{1}{\rho}\frac{d\rho}{dp}\operatorname{grad}p = \frac{1}{c^2}\operatorname{grad}P$

and $\operatorname{grad}P$ obtained from (2b). Then we have

$$(9) \qquad \operatorname{div}\mathfrak{W} - \frac{1}{c^2}\mathfrak{W}\cdot\operatorname{grad}\frac{\mathfrak{W}^2}{2} = 0.$$

But $\mathfrak{W}\cdot\operatorname{grad}\frac{\mathfrak{W}^2}{2}$ can be written in the rectangular velocity components u, v & w. Then

$$\mathfrak{W}\cdot\operatorname{grad}\frac{\mathfrak{W}^2}{2} = \mathfrak{W}\cdot\nabla\mathfrak{W}\cdot\mathfrak{W} = u^2\frac{\partial u}{\partial x} + v^2\frac{\partial v}{\partial y} + w^2\frac{\partial w}{\partial z}$$

$$+ uv\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + vw\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + wu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right).$$

Then equation (9) becomes, with this coordinates

$$(9a) \qquad \frac{\partial u}{\partial x}\left(1 - \frac{u^2}{c^2}\right) + \frac{\partial v}{\partial y}\left(1 - \frac{v^2}{c^2}\right) + \frac{\partial w}{\partial z}\left(1 - \frac{w^2}{c^2}\right)$$

$$- \frac{2}{c^2}\left(uv\frac{\partial u}{\partial y} + vw\frac{\partial w}{\partial x} + wu\frac{\partial w}{\partial x}\right) = 0.$$

(7)

(In this equation, we have make use of the condition that the flow is irrotational, so $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$, etc.)

From the construction of Eq.(9), it can be easily seen that, when the velocity components, u, v, w are everywhere small compared with the sound velocity, then the equation will be reduced to $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ or $\operatorname{div} \vec{v} = 0$. When the resultant velocity is not so small compared with the sound velocity as to be neglected, but smaller than the sound velocity, then although the quantitative pattern of the flow will vary, but the general character must be still similar to the incompressible fluid. When the flow velocity is of the same order as the sound velocity, we can still choose such a coordinate system, that the x-axis falls with the general direction of velocity, so only u is large in comparison with c; other components v, w will be negligible with respect to c, so the equation (9) reduces to

$$(10) \qquad \frac{\partial u}{\partial x}\left(1 - \frac{u^2}{c^2}\right) + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

When there is a pressure drop in the $x+$ direction, the velocity u will be increased. Therefore $\frac{\partial u}{\partial x}$ is positive. Then from Eq.(10), it can be easily seen that $\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ is negative if $u^2 < c^2$, it is positive

when $u^2 > c^2$; it is zero when $u^2 = c^2$. In (8)
other words, if the flow velocity is increasing, then
the flow is converging when the velocity is smaller
than the sound velocity; it is diverging when the
velocity greater than the velocity of sound; and
the crosssection of flow remains constant when the
velocity is just equal to the sound velocity. All these
phenomena can be observed from the flow in
a Laval nozzle. When the pressure in the nozzle
is all the way decreasing, the velocity in the
convergent part is below the sound velocity; and
that in the diverging part is above sound velocity.
The velocity is just equal to the sound velocity in the
minimum cross-section.

The Eq. (10) can be further changed by using Eq. (1),

$$(10a)\qquad \frac{\partial^2\Phi}{\partial x^2}\left(1-\frac{u^2}{c^2}\right)+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0.$$

For the approximate treatment of flow in the near
environment of a point, we can put u equal to
the mean value in this region. Then we get a
tremendous simplification of the differential equation
(10a). If the value of $1-\frac{u^2}{c^2}$ is positive, then the
equation is so called elliptical type, which is used
throughout the incompressible fluid. When the value of

⑨

$1-\frac{u^2}{c^2}$ is negative, ($u>c$) the differential equation is of hyperbolic type. We know that a equation of elliptical type always give solution that is regular in within the region; while a equation of hyperbolic type will gives discontinuous solution in the region. The discontinuity run across the region along the so-called characteristics of the differential equation. The typical character of both types of equation can be illustrated clearly in the corresponding case of the singularity of a source in the incompressible fluid $\left(\Phi=\frac{A}{r}=\frac{A}{\sqrt{x^2+y^2+z^2}}\right)$.

The problem is to find the potential of flow generated by a source superposed by a ~~constant~~ velocity of same order of sound velocity. The problem can be simplified if we assumed the velocity caused from the source be small, so that eq. (10) or (10a) can be used. To find the singularity of this case, we choose such a reference system, that the undisturbed medium is at rest, & the source itself moves with a ~~constant~~ velocity $-u_0$. We then consider a "explosion wave" of spherical form & propagate along all directions. We can get the potential of such a wave by adding a function f to Eq. (7), which only in a short interval is different from zero, beyond this interval it vanishes.

This corresponds to the case that a small sound source suddenly enlarge ~~begin to~~ & then keep its new value. We can then assume a continuous process of this short expansions, and the centre of these expansions moves with a velocity $-u_0$ in the space. Since now we are dealing the propagation of sound, we can use eqn. (5), and due to the linearity of the differential equation, we can obtain the potential by summation of the potentials of each explosion wave. The derivation will be omitted. but after a long time if the instantaneous centre of expansion is at the origin of the coordinate system, then we have the formula (10)

$$(11) \qquad \phi = \frac{A}{\sqrt{x^2 + \left(1 - \frac{u_0^2}{c^2}\right)(y^2+z^2)}}.$$

When the velocity u_0 is smaller than c, then the surfaces of constant potential are flattened ellipsoids of rotation, instead of spheres which is the case of incompressible fluid. The near the u_0 to c, more flatten the ellipsoid. When u_0 is greater than c, the solution is different from zero only within the cone of semi-apex angle α.

$$\tan\alpha = \pm \frac{1}{\sqrt{\frac{u_0^2}{c^2} - 1}}$$

(12). Or $\sin\alpha = \pm \frac{c}{u_0}$.

We can also arrive at the same result if we investigate the instantaneous location of the explosion wave, as Fig. 1



Fig. 1

The angle α is called Mach's angle. (We can remark here, that in case $u_0 < c$ the explosion wave fills the whole space in all direction; in case $u_0 > c$, only within the cone). The surfaces of constant potential for the case of $u_0 > c$, is hyperboloids of two sheets with the cone as asymptotic cone. However only one sheet of the hyperboloid has physical meaning due to the fact that potential is produced by the explosion wave. In the case that we built up such potentials of source flow to give other potentials, we should bear in mind that outside the given cone, the potential of each source is zero & only with the cone the potential has the value given by Eq. (11).

3. Flow with subsonic velocity — Lorentz's theory. (12)

The formula (11) gives us a suggestion to find a general solution of the equation (10)a. For every case of subsonic velocity, we can solve the equation like the case of incompressible fluid, where

$$\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} + \frac{\partial^2 \phi}{\partial \zeta^2} = 0,$$

and connect the two by the relation

$$(13) \qquad \xi = x; \quad \eta = y\sqrt{1-\frac{u_0^2}{c^2}}, \quad \zeta = z\sqrt{1-\frac{u_0^2}{c^2}}.$$

so that at the corresponding points of the two space, we have the same potential. (2) Of course, we should remember that the additional velocity from the potential should be small in comparison with the ground velocity u_0. Therefore this solution cannot be applied to flow with a stagnation point, since there the deviation from u_0 is very large. However, for bodies start with a pointed nose or edge, this method should yield valuable results. When we know a potential of incompressible fluid over a body, our problem is to find how the pattern will change when we go to compressible fluid by this method. The slope $\frac{dy}{dx} = \frac{v}{u} \simeq \frac{1}{u_0}\frac{\partial \phi}{\partial y}$.

Correspondingly $\quad \frac{dy}{d\xi} = \frac{1}{u_0}\frac{\partial\varphi}{\partial\eta}$; and $\frac{\partial\phi}{\partial y} = \frac{\partial\varphi}{\partial\eta}\sqrt{1-\frac{u_0^2}{c^2}}$. (13)

we have
$$\frac{dy}{dx} = \frac{d\eta}{d\xi}\sqrt{1-\frac{u_0^2}{c^2}}.$$

Although the points on the wall of body is not transformed by the formula, but by increasing the distance in y-direction so that we have the same potential, nevertheless, the difference is of higher order, and we can say that the whole picture is shortened by the ratio $\sqrt{1-\frac{u_0^2}{c^2}}$, when we have the same value of potential as the incompressible fluid; & also the angle of attack must be decreased in the same ratio.

We can, of course, say that the potential Φ in the x-y-z space is increased by so many times over that in the ξ-η-ζ space. If we choose the ratio of potential to be $1/\sqrt{1-\frac{u_0^2}{c^2}}$, so $\frac{dy}{dx} = \frac{\partial\eta}{\partial\xi}$, i.e., the profile & angle of attack coincide each other. The difference in pressure will be then $1/\sqrt{1-\frac{u_0^2}{c^2}}$ times stronger, & so the process of separation is hastened.

It is also interesting to investigate the pressure distribution over the surface. According to the Bernoulli's equation, this is controlled by $\rho u \frac{\partial u}{\partial x}$. Here it is equal to $\rho u_0 \frac{\partial^2 \phi}{\partial x^2}$. Since we have set $\xi = x$, & the potentials at corresponding points in these two space are the same; therefore this transformation also yields same pressure at corresponding points. Hence we should expect that the possible separation process in both spaces is also the same. Therefore we can draw the conclusion that to avoid separation, the profile must be made flatter, & the angle of attack smaller, the nearer the basic velocity "u" approaches the sound velocity. Based on this approximate theory, the maximum lift coefficient that can be reached by the profile is same in compressible as in incompressible fluid. The actual case, this theory approximates the truth poorer and poorer when we approach the velocity of sound. The chief reason of this discrepancy is that the additional velocity is not small compared with the fundamental velocity, and that when the basic velocity is near to the sound velocity, the velocity of flow over the suction side may equal or

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Taylor's Electrical analogy

The relations for a compressible fluid are

$$\frac{\partial\varphi}{\partial x} = -u, \quad \frac{\partial\varphi}{\partial y} = -v, \quad \frac{\partial\psi}{\partial x} = \rho v, \quad \frac{\partial\psi}{\partial y} = -\rho u \qquad (5)$$

The relations for a electrical field is

$$\frac{\partial V}{\partial x} = -f\sigma, \quad \frac{\partial V}{\partial y} = -g\sigma, \quad \frac{\partial W}{\partial x} = tg, \quad \frac{\partial W}{\partial y} = -tf \qquad (6)$$

where V is the electrical potential, W = electrical stream function, f, g = components of current density, σ = specific resistance, t = depth of electrolyte.

There are two possible analogy between (5) & (6),

(A) $V=\varphi$, $W=\psi$, $u=f\sigma$, $\rho v = tg$; $v = g\sigma$; $\rho u = tf$ (7)

so $t=\rho\sigma$; Or

(B) $W=\varphi$, $V=\psi$, $-u=tg$, $\rho v = f\sigma$; $v = tf$; $\rho u = -g\sigma$ (8)

so $t=\sigma/\rho$.

In the first analogy (A), the depth of electrolyte is proportional to the density of the fluid flow, and the direction of flow is same as the max. current density. Since the velocity normal to any solid body is zero on the surface, so the body must be represented by a non-conductor in the electrical field. In analogy (B), the depth of electrolyte is inversely proportional to ρ, and

the velocity is perpendicular to the direction of maximum current density. Therefore the solid body must be represented by a perfect conductor in the electrolyte.

The procedure used in this instrument, is first make a electrolyte bath of constant depth, & after finding out the values of $\theta = q/v$, where $q^2 = \left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2$ we can use the Bernoulli's equation to find out the ρ/ρ_0 when ρ_0 = density at infinity;

$$\frac{\rho}{\rho_0} = \left[1 - \frac{1}{2}(\gamma-1)n^2(\theta^2-1)\right]^{\frac{1}{\gamma-1}} \tag{9}$$

where $v = na$, & $\gamma = 1.40$. Knowing this ratio we can remodel the bath to corresponding variable depth, and a second approximation can be made.

The equation of Rayleigh's first approximation is

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = -\frac{1}{\rho}\left(\frac{\partial\rho_1}{\partial x}\frac{\partial\varphi_1}{\partial x} + \frac{\partial\rho_1}{\partial y}\frac{\partial\varphi_1}{\partial y}\right) \tag{10}$$

The equation of electric analogy is

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = -\frac{1}{\rho}\left(\frac{\partial\rho_1}{\partial x}\frac{\partial\varphi}{\partial x} + \frac{\partial\rho_1}{\partial y}\frac{\partial\varphi}{\partial y}\right) \tag{11}$$

So the difference is only in the value of φ in the right hand side of the equation.

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Due to the fact that electrical potential is single valued function, so when the flow to be approximated involves circulation as the flow over an airfoil, only analogy (B) can be used. Then the electrical stream function will be multi-valued one.

The result of investigation of flow around a cylinder by analogy (A) shows that the process is convergent for $v/a_0 < 0.45$ & divergent when $v/a_0 > 0.4$[illegible]. In case of airfoil by analogy (B), the process is convergent for $v/a_0 < 0.58$, & divergent for $v/a_0 > 0.5$[illegible]. However both cases show the fact that the critical condition is that the highest velocity in the field exceeds the local sound velocity.

1.1.2.2
高速气流突变之测定

这是发表于1941年的“高速气流突变之测定”（A Method for Predicting the Compressibility Burble）一文的部分手稿以及计算图表，共有13页。

1940年由于王助的推荐，作者成为成都航空研究所的通讯研究员，写了这篇专论，形成该所的第二号研究报告。

当飞行速度逐渐增加到某一“临界速度”时，气流会发生突然的变化。这时，气流中某些地方的流速达到局部声速，导致飞行体所受的阻力骤然巨增。可见，求取临界速度对于高速飞机的设计甚为重要。这一临界速度固然可以用风洞试验去实测，但耗资巨大；更方便的是用作者1939年发表的计算高亚声速流动的近似方法通过计算来解决。

这篇文章的第1部分首先介绍了该近似算法。第2部分应用该法计算了圆柱绕流问题的临界速度，并和用摄动法算得的结果做了比较。作者发现，要将这一算法应用于实际问题时，原则上需要对飞行体（如机翼）的形状做出修正。这项工作的计算量一般是很大的。作者为了寻求简化计算的途径，在第3部分专门就此问题做了分析，说明即使不作形状修正也不至于带来显著的误差；接着，作者对圆柱绕流问题重新做了简化计算，并将计算和实验数据做了对比，发现两者相当一致，从而说明简化方法是成功的。

A Method of Predicting the Critical Speed for Compressibility Burble

Summary

In Part I of the present report, a method of calculating the two dimensional subsonic compressible flow is given based upon the idea of using the tangent to the adiabatic pressure-volume curve as an approximation to the curve it itself. In Part II, this method is applied to the case of flow over a circular cylinder with its axis perpendicular to the general flow direction. The result is compared with that of perturbation method. In Part III, the conclusion drawn from Part I & Part II is used to develope a procedure which enables the prediction of the critical speed for compressibility burble from test data at low speeds. Then the result of method is compared with those obtained from Jacobs' methods and the experimental data. It is found that the new method gives better agreement with experimental data.

The experimental data obtained from high-speed wind tunnel shows that when the maximum velocity of fluid over a solid body reaches the local velocity of propagation of sound, the resistance experienced by the body suddenly increases. This phenomena is generally called compressibility burble. The speed at which compressibility burble occurs is called the critical speed of the body. Therefore in the design of modern high-speed aircraft, the designer should avoid the compressibility burble as far as he can. Unfortunately, to determine critical speed of a body experimentally requires a costly high speed wind tunnel.

Hence it is desirable to have a reliable method to calculate the critical speed either theoretically or from experimental data obtained from an ordinary low speed wind tunnel.

The calculation of the critical speed of a body requires, of course, the solution of the problem of compressible flow over the body. The existing methods devised for this purpose are

(1) Prandtl-Glauert method

(2) Perturbation method

and (3) Hodograph method.

The theory of Glauert and Prandtl is based on the assumption that the disturbance produced by the body placed in a parallel flow is small. They are thus able to linearize the partial differential equation for the velocity potential and obtain a very simple solution. It is evident that the theory can be applied only to a very thin airfoil or very slender body, because only then the disturbance produced by the body is small. But even for this case, the theory breaks down for points near the stagnation point. For bodies which are generally seen in aircraft engineering, this method gives a higher critical speed than that experimentally observed. In other words, the Glauert-Prandtl theory is not conservative.

The perturbation method is developed by Lord Rayleigh, O. Janzen, L. Poggi, and others. It consists essentially of expanding the velocity potential into a series of ascending powers of M_1^2 where M_1 = ratio of the velocity of undisturbed parallel flow to the velocity of sound in the parallel flow. M_1 is generally called the Mach's number. This method is theoretically exact, if the convergence of the series can be established. However, the practical calculation is very involved even for simple shaped body if one goes beyond the first approximation (i.e., M^2 term).

The hodograph method is first suggested by Molenbroek & Chaplygin. In this method, the inclination of the velocity vector to a fixed line and the magnitude of the velocity is used as the independent variables. It is a particular application of Legendre's contact transformation. The main drawback of this method is the difficulty of determine the solution by the boundary condition. Therefore this method although exact and elegant is applied till now only to few isolated cases.

The method used in the present report is first suggested to the author by Dr. Th. v. Kármán. Its general theory is discussed in a previous paper (Ref. 1). In following paragraphs a review of this theory will be first given.

Part (I)

It is well-known that if the pressure in the fluid can be expressed as a function of density of the fluid only, the flow of compressible fluid is irrotational. Then there exists a potential velocity ϕ defined as

$$d\phi = u\,dx + v\,dy$$

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y} \qquad (1)$$

For two dimensional case, the condition of zero vorticity is expressed by

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \qquad (2)$$

where u and v are the components of velocity in the x and y directions. The condition for the conservation of mass is expressed by the equation of continuity which in case of steady motion can be written as

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad (3)$$

Eq. (3) can be automatically satisfied by introducing so-called stream function ψ, defined as

$$d\psi = \frac{\rho}{\rho_0}(u\,dy - v\,dx)$$

$$\frac{\partial \psi}{\partial y} = \frac{\rho}{\rho_0} u, \qquad -\frac{\partial \psi}{\partial x} = \frac{\rho}{\rho_0} v \qquad (4)$$

where ρ is the density and the subscript 0 indicates the value of the variable at the state when $u = v = 0$. The velocity potential ϕ and stream function ψ is here introduced to satisfy the kinematical properties of the fluid field. The dynamical relation of the flow in this special case of irrotational steady flow is expressed by the generalized Bernoulli's equation.

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$$\frac{1}{2}u^2 + \int_0^p \frac{dp}{\rho} = \text{constant.} \tag{5}$$

The Eqs. (1), (3) (4) represent 3 equations for 5 unknowns u, v, p, ρ, & T, where T is the temperature of the fluid. Therefore for complete solution of the problem, two more equations are necessary. The two equations are the equation of state of the fluid and the equation of total energy of the fluid. For perfect gas, the equation of state is

$$\frac{p}{\rho} = RT \tag{6}$$

where R is the gas constant. If no heat is added & removed from the fluid in the whole field, then the condition of constant total energy is expressed by the adiabatic law

$$\frac{p}{\rho^\gamma} = \text{constant} \tag{7}$$

where γ is the ratio of specific heat of the fluid at constant pressure c_p and that at constant volume c_v. In other words,

$$\gamma = \frac{c_p}{c_v} \tag{8}$$

The value of γ for diatomic gas such as air is 1.405.

Therefore the fluid problem can be solved completely by means Eqs (1), (3), (6), (6) & (7), or its equivalent Eqs. (3), (4) (5) (6) & (7).

Now suppose, a solution of this system of equation is found for the flow around a certain body. Then the question arises: Whether this known solution can be utilized to construct another solution which satisfies the same type of equations as (1), (3), (5) but perhaps different equation of state & of the conservation of energy! The difference in the equation of state & of conservation of energy means only the difference in the properties of the fluids in the two solutions. To show this, two new functions X, Y have to be introduced such that

the parallel flow. To find the velocity and pressure distribution over a given body at different Mach's number of the parallel flow it is thus necessary to start with different solution corresponding to different body shapes so that in the plane of compressible fluid, the given shape of the body is obtained. This introduces some complexity. However, the labor involved is, perhaps, much less than the perturbation method developed by Lord Rayleigh, O. Janzen + others.

Part II

As stated previously, the only approximation involved in the present theory is the substitution of the tangent to the adiabatic curve for the adiabatic curve itself. It thus becomes necessary to check the amount of error introduced by this approximation. The only two-dimensional subsonic flow calculated with the true adiabatic law to sufficient accuracy is that around a circular cylinder placed with its axis perpendicular to the general direction of the parallel stream. The problem is calculated by means of perturbation method up to terms of order M_0^2 by I. Imai,[14] K. Tamada + Y. Saito.[14] The theory developed in Part (I) will be presently applied to calculate the flow over a circular cylinder and the result compared with that obtained by the authors mentioned.

The transformation from incompressible flow to compressible flow involves a slight distortion of the shape of the body.[31] The main effect is found to be a slight increase in the ratio of the maximum thickness to the length of the body. Therefore in order to obtain a circular section after the transformation, it is necessary to start with an elliptic section with its major axis in the direction of uniform flow. The incompressible flow over an elliptic cylinder can be obtained by applying the well-known Joukowsky's transformation to the flow over a circular cylinder in the

complex coordinate ζ with the center of the circle located at the origin of the ζ-plane (Fig. 3)

Hence

$$G(\zeta) = \zeta + \frac{b^2}{\zeta}$$

$$\bar{G}(\bar\zeta) = \bar\zeta + \frac{b^2}{\bar\zeta} \qquad (52)$$

where b is the radius of the circle in the ζ-plane. The radius b controls the thickness ratio of the elliptic section [illegible] the thickness ratio [illegible] $b = 1$ [illegible]. The Joukowsky transformation is

$$z = \zeta + \frac{1}{\zeta} \qquad (53)$$

thickness ratio is [illegible] the [illegible]

It is convenient to carry out the computation by using the ζ coordinate. Then Eq (51) can be written as

$$Z = z - \frac{M_1^2}{(1+\sqrt{1-M_1^2})^2}\int \frac{\left(\frac{d\bar G}{d\bar\zeta}\right)^2}{\left(\frac{d\bar z}{d\bar\zeta}\right)^2\left(\frac{dz}{d\zeta}\right)} d\zeta \qquad (54)$$

Substituting Eqs (52) & (53) into Eq. (54), the complex coordinate in the compressible flow is obtained as

$$Z = z - \frac{M_1^2}{(1+\sqrt{1-M_1^2})^2}\int \left(1 - \frac{b^2}{\bar\zeta^2}\right)^2 \frac{1}{1-\frac{1}{\bar\zeta^2}}\, d\bar\zeta$$

$$= z - \frac{M_1^2}{(1+\sqrt{1-M_1^2})^2}\left\{\bar\zeta + \frac{b^4}{\bar\zeta} + \frac{1}{2}(b^2-1)^2 \log\frac{\bar\zeta-1}{\bar\zeta+1}\right\} \qquad (55)$$

In order to find the distortion of the shape of the original elliptic section by the transformation,

$$\zeta = b e^{i\theta} \; ; \quad \bar\zeta = b e^{-i\theta} \qquad (56)$$

Substituting Eq (56) into (55) & separating real & imaginary parts,

$$X = \left(b + \frac{1}{b}\right)\cos\theta - \frac{M_1^2}{(1+\sqrt{1-M_1^2})^2}\left[b(b^2+1)\cos\theta - \frac{(b^2-1)^2}{4}\log\frac{1+\frac{2b}{b^2+1}\cos\theta}{1-\frac{2b}{b^2+1}\cos\theta}\right]$$

$$Y = \left(b - \frac{1}{b}\right)\sin\theta - \frac{M_1^2}{(1+\sqrt{1-M_1^2})^2}\left[b(b^2-1)\sin\theta - \frac{(b^2-1)^2}{2}\tan^{-1}\frac{2b\sin\theta}{b^2-1}\right] \qquad (57)$$

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By setting $\theta = 0$ and $\frac{\pi}{2}$ in the first & the second expression of Eq. (54) respectively, the major & minor axis of the approximately elliptic section in compressible flow are obtained. For a circular section the ratio is equal to the ratio of the minor axis to the major axis is equal to the thickness ratio of the section. Thus the condition that the section in compressible flow shall be appropriately circular is

$$1 = \frac{b^2-1}{b^2+1}\;\frac{1+\frac{M_1^2}{(1+\sqrt{1-M_1^2})^2}\left\{b^2 - \frac{b(b^2-1)}{2}\tan^{-1}\frac{2b}{b^2-1}\right\}}{1-\frac{M_1^2}{(1+\sqrt{1-M_1^2})^2}\left\{b^2-\frac{b(b^2-1)}{2}\left(\frac{b^2-1}{b^2+1}\right)\log\frac{b+1}{b-1}\right\}} \qquad (58)$$

For a given Mach number M_1 of the parallel stream, the value of b can be solved numerically. For example, for $M_1 = 0.400$, b is found to be 5.7637.

When the value of b is thus determined, the detailed shape of the section in compressible flow can be calculated by means of Eq. (54). For the case $M_1 = 0.400$, the section obtained by the transformation with $b = 5.7637$ is shown in following Table, where $r^2 = x^2 + y^2$; $\phi = \tan^{-1}\frac{y}{x}$.

b	r	ϕ

If the section in compressible were a true circle, r should be a constant. The variation of r in the above table is only 0.03%. This variation is, indeed, negligible. Therefore the velocity & pressure distribution over this particular section can be considered as that over a true circular section.

By putting $\zeta = i\eta$ in Eq. (55), the coordinate of a point along the transverse axis of symmetry can be obtained as

$$Y = \left(\eta - \frac{1}{\eta}\right) - 2\left[\eta\left(\frac{b^4}{\eta^2} - 1\right) - \frac{1}{2}(b^2-1)^2 \tan^{-1}\frac{2\eta}{\eta^2 - 1}\right] \qquad (59)$$

The velocity over the ellipse at the corresponding point in the incompressible flow is

$$\frac{\frac{dG}{d\zeta}}{\frac{dz}{d\zeta}} = \frac{\eta^2 + b^2}{\eta^2 + 1} \qquad (60)$$

Hence the corresponding velocity in the compressible flow can be calculated by means Eq. (40), and expressed as

$$\frac{W}{W_1} = \frac{\eta^2 + b^2}{\eta^2 + 1} \cdot \frac{1 - \frac{M_1^2}{(1+\sqrt{1-M_1^2})^2}}{1 - \frac{M_1^2}{(1+\sqrt{1-M_1^2})^2}\left(\frac{\eta^2+b^2}{\eta^2+1}\right)^2} \qquad (61)$$

The results of this calculation for the case $M_1 = 0.400$, using the previous determined appropriate value of $b = 5.7827$, is shown in Fig. 4 (1). The ordinate is the increase in velocity ΔW due to compressibility divided by the velocity W_1 of the parallel stream. The abscissa is the ratio of the distance Y from the point concerned to the center of the circular section and the radius of the section, R.

According to Lord Rayleigh & O. Janzen, the perturbation method gives as first approximation

$$\frac{\Delta W}{W_1} = \left\{\frac{11}{6}\left(\frac{Y}{R}\right)^{-2} - \frac{3}{4}\left(\frac{Y}{R}\right)^{-4} + \frac{1}{12}\left(\frac{Y}{R}\right)^{-6}\right\} M_1^2 \qquad (62)$$

which is independent upon the ratio of specific heats γ. (Ref. 3)
The second approximation of perturbation method is carried out by I. Imai, K. Tamada & Y. Saito (Ref. 4)

$$\frac{\Delta W}{W_1} = \left\{\frac{11}{6}\left(\frac{Y}{R}\right)^{-2} - \frac{3}{4}\left(\frac{Y}{R}\right)^{-4} + \frac{1}{12}\left(\frac{Y}{R}\right)^{-6}\right\} M_1^2 + (\gamma - 1)\left\{\frac{12}{60}\left(\frac{Y}{R}\right)^{-2} + \frac{19}{20}\left(\frac{Y}{R}\right)^{-4} - \frac{3}{5}\left(\frac{Y}{R}\right)^{-6} + \frac{1}{80}\left(\frac{Y}{R}\right)^{-8} + \frac{1}{80}\left(\frac{Y}{R}\right)^{-10}\right\} M_1^4$$
$$+ \left\{\frac{257}{80}\left(\frac{Y}{R}\right)^{-2} - \frac{17}{24}\left(\frac{Y}{R}\right)^{-6} - \frac{3}{4}\left(\frac{Y}{R}\right)^{-8} + \frac{1}{40}\left(\frac{Y}{R}\right)^{-10}\right\} M_1^4 \qquad (63)$$

the values of $\frac{\Delta W}{W_1}$ calculated from both Eq. (62) & Eq. (63) are shown in Fig. 4. It is seen that the theory developed in Part I of the present report gives a value of velocity increment $\frac{\Delta W}{W_1}$ between the first & the second approximation of the perturbation theory with $\gamma = 1.405$. Unfortunately, it is not certain whether exact solution with $\gamma = 1.405$ will further increase the value of $\frac{\Delta W}{W_1}$ or decrease it. Therefore it is impossible at present to ascertain the exact magnitude of deviation introduced by using the tangent to the adiabatic curve instead of the adiabatic curve itself.

By comparing the approximate adiabatic relation given by Eq. (18) with the true adiabatic relation given by Eq. (8), it is seen that the relation given by Eq. (18) can be also considered as the exact adiabatic relation of a fictitious gas with $\gamma = -1$. Therefore the solution given by the present theory can be also considered as exact solution of the flow of this fictitious gas. Hence by putting $\gamma = -1$ in Eq. (63), and comparing the value of $\frac{\Delta W}{W_1}$ so calculated with that obtained by present theory, i.e. the error involved in the second approximation can be determined. This comparison is shown in Fig. 4. The curve of exact solution for $\gamma = -1$ is slightly higher than that of the second approximation of perturbation theory with $\gamma = -1$. If this tendency of increasing the value of $\frac{\Delta W}{W_1}$ also holds in case of $\gamma = 1.405$, then the value of velocity increment $\frac{\Delta W}{W_1}$ given by the straight-line adiabatic relation being lower than the second approximation for $\gamma = 1.405$ will be definitely lower than that of the exact solution. Taking this lower value of velocity increment as the true value, to calculate the critical Mach number of the airfoil, the resulting value of the critical Mach number will be higher than it should be. In other words, the method is slightly unsafe.

PART III.

The most difficult part of the calculation in applying the theory developed in part I to practical cases is the correction for the shape of the body. Because the correction formula, Eq (57), involves the complex potential of a corresponding incompressible flow. Although the principle for finding this complex potential for any given shape of the body in uniform flow is well-known, the actual calculation is generally very tedious, except for simple geometrical shapes. The problem then arises: whether it is possible to omit this correction without introducing appreciable error in the velocity and the pressure distribution. Furthermore as shown by the calculation for elliptical cylinders, the correction for the shape of body results a slight increase in the thickness ratio of the section. An increase in thickness ratio of the section generally raises the maximum local speed over the surface of the section. In view of the previously shown result that the method developed in Part I, slightly underestimates the velocity in compressible flow, this increase in velocity due to neglect of the correction in shape will tends to remedy this defect.

To test whether this reasoning is justified, the flow over an circular cylinder will be investigated anew. However, now the slight change in shape of the body in transforming from incompressible flow to compressible flow will be neglected.* The complex potential for incompressible flow over an circular cylinder is given by

$$\phi + i\psi = w_1 \left[z + \frac{b^2}{z} \right] \qquad (64)$$

where b is the radius of the cylinder. By differentiating Eq (64), the velocity at a point

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* This calculation is incidentally carried out by K. Tamada (Ref. 8)

corresponding to $\zeta = i\eta$ can be formed as

using Eq (64)

$$\frac{W}{W_1} = 1 + \frac{b^2}{\eta^2} \qquad (65)$$

Therefore the velocity in compressible flow is

$$\frac{W}{W_1} = \left(1 + \frac{b^2}{\eta^2}\right) \frac{1 - \frac{M_1^2}{[1+\sqrt{1-M_1^2}]^2}}{1 - \frac{M_1^2}{[1+\sqrt{1-M_1^2}]^2}\left(1 + \frac{b^2}{\eta^2}\right)^2} \qquad (66)$$

Substituting Eq (64) into Eq (57), the coördinate of a point in compressible flow can be expressed in terms of the coördinate of the corresponding point in incompressible flow. Thus

$$Z = z - \frac{M_1^2}{[1+\sqrt{1-M_1^2}]^2}\left(\bar{z} + \frac{2b^2}{z} - \frac{b^4}{3\bar{z}^3}\right) \qquad (67)$$

By putting $z = \bar{z} = b$ in Eq (67) the axis of the transformed parallel to the uniform flow is found to be

$$b\left(1 - \frac{2}{3}k\right) = \qquad (68)$$

By putting $z = ib$ & $\bar{z} = -ib$ in Eq. (67), the axis of the transformed section perpendicular to the uniform flow is found to be

$$b\left(1 - \frac{2}{3}k\right) = a = \qquad (69)$$

Therefore the thickness ratio of the transformed section is instead of 1.00. By letting $z = i\eta$, $\bar{z} = -i\eta$ in Eq. (67), the point on the vertical axis of symmetry can be calculated as

$$Y = (i\eta) + \frac{M_1^2}{[1+\sqrt{1-M_1^2}]^2}\left[\eta - \frac{2b^2}{\eta} - \frac{b^4}{3\eta^3}\right] \qquad (70)$$

Using Eqs (66) & (70), the velocity distribution along the vertical axis of symmetry can be calculated. The result of the calculation is shown in Fig. 6.

By comparing with the curves for the second approximation of the perturbation theory it is clear that the new method gives a higher velocity over the surface of the

section. Therefore if this value is used to predict the critical velocity of the section, the result will be slightly conservative which is very desirable from engineering stand point. The circular section chosen for illustrating the method is, however, an extreme case in the sense that the velocity increment is very large. For airfoils generally used in design, the thickness ratio is much smaller, and therefore the velocity increment will be also much smaller. Then the discrepancy between the present simplified theory and the exact solution will be much less than that shown in Fig. 6. Therefore, the neglect of the correction in body shape as given by Eq. (67) will not introduce serious error, but on the other hand will make the method slightly conservative and greatly reduce the amount of calculation.

Summarizing, if the velocity and the pressure distribution over a body is measured at low speed in ordinary wind tunnel, the velocity and the pressure distribution over the same body at high speeds can be calculated by means of Eq. (46) and (45).* Table I and II ... By integrating the pressure distribution over the surface of the airfoil, the change in lift and moment coefficients of an airfoil with change in Mach's number can be obtained. It is interesting to notice that Eq. (45) is reduced to the relation of Glauert

$$C_p = \frac{c_p}{\sqrt{1-M^2}} \qquad (71)$$

if c_p is very small and c_p^2 can be neglected. However, for suction pressures, the present method gives larger increase due to compressibility than Glauert's method.

To calculate the critical speed of a body when its maximum suction pressure is given, the maximum velocity in incompressible flow has to be calculated first. Using Eq. (46) ...

* It is mentioned by Th. von Kármán during a recent lecture below that the result of calculation by this method agrees well with recent NACA (U.S.A.) tests. However due to the confidential nature of those tests, he is not allowed to reproduce them.

(21), the following relation is obtained: 21

$$\frac{W_{MAX}}{W_1} = (1 - C_{p_{MAX}})^{\frac{1}{2}} \frac{1 - \frac{M_c^2}{(1+\sqrt{1-M_c^2})^2}}{1 - \frac{M_c^2}{(1+\sqrt{1-M_c^2})^2}(1 - C_{p_{MAX}})} \qquad (22)$$

However, according to Kaplan (Ref. 6) and others, the maximum local velocity and critical speed are connected by the following relation

$$\left(\frac{W_{MAX}}{W_1}\right)^2 = \frac{2}{\gamma+1}\frac{1}{M_c^2} + \frac{\gamma-1}{\gamma+1} \qquad (23)$$

where M_c is the critical Mach's number, i.e., the Mach's number of uniform flow when compressibility bubble exists. Equating the two expressions proposed by (22) and (23), the equation for computing the critical Mach's number is

$$(1 - C_{p_{MAX}})^{\frac{1}{2}} \frac{1 - \frac{M_c^2}{(1+\sqrt{1-M_c^2})^2}}{1 - \frac{M_c^2}{(1+\sqrt{1-M_c^2})^2}(1 - C_{p_{MAX}})} = \frac{2}{\gamma+1}\frac{1}{M_c^2} + \frac{\gamma-1}{\gamma+1} \qquad (24)$$

The result of the calculation is shown in Fig. 7, together with Jacobs' the curve given by ... The experimental data given by J. Stack etc. Ref. (13) is also included for comparison. It is seen that the present method gives better agreement with the experimental data than Jacobs' method.

Appendix I.

APPENDIX II

Velocity of sound in Standard Atmosphere

$$A = 1130\sqrt{\frac{519 - 0.003566\,h}{519}} \text{ feet/sec.}$$

$$= 763\sqrt{\frac{519 - 0.003566\,h}{519}} \text{ m.p.h.}$$

where h = altitude in feet. Or

$$A = 1229\sqrt{\frac{288 - 0.0065\,H}{288}} \text{ km.p.h.}$$

where H = altitude in meters.

1.1.3

Superaerodynamics – Mechanics of Rarefied Gases
超级空气动力学——稀薄气体力学

这是发表于 1946 年 “Superaerodynamics—Mechanics of Rarefied Gases”（超级空气动力学——稀薄气体力学）一文的部分手稿，包括初稿的一部分，以及关于低密度和自由分子流的推导和算表等，共有 14 页。

超级空气动力学——稀薄气体力学（Superaerodynamics）这个学科最早是由 Zahm 提出来的，他在 1934 年发表了一篇论文，讨论有关高度稀薄气体的动力学问题。限于当时喷气技术的水平，要在那么高的高空飞行几乎是不可能的，所以超级空气动力学——稀薄气体力学（Superaerodynamics）只是一个纯学术研究的课题，并无实际的工程意义。到了 40 年代中期，作者考虑到喷气推进技术已经有了长足的进步，飞机的飞行不应受到高度的限制。远程喷气飞机（rocket airplane）的最优飞行高度估计在 100km 左右，那里的空气已经非常稀薄，不能当作常规流体力学中的连续介质看待，必须运用超级空气动力学——稀薄气体力学（Superaerodynamics）的概念和方法来指导飞机的设计。除了高空远航以外，超级空气动力学——稀薄气体力学（Superaerodynamics）的知识还可应用到很多涉及低密度的工业过程，例如用于蒸馏和其他化工作业的高真空泵的设计。作者写了这篇论文，来讨论这一流体力学新分支的基本概念和说明某些已经得到的结果，以便引起大家的重视，推动这一流体力学分支的发展。后来，“Superaerodynamics”（超级空气动力学）一词很少有人采用，人们普遍采用 “Mechanics of Rarefied Gases”（稀薄气体力学）一词。

作者在这篇文章中首先介绍了分子运动平均自由程 l 的概念，并用 l 与物体的特征长度 L（或边界层厚度 δ）之比 l/ L（或 l/ δ）形成一个无量纲常数，在由马赫数 M 和雷诺数 Re 构成的平面上，以 l/ δ 为指标把该平面划分为四个区域，即：自由分子流区、过渡区（其特征是分子间的碰撞和分

子与物体表面的碰撞同等重要）、滑流区和气体动力学区。于是，不同的流动问题可以由 M 数和 Re 数两个数值来判断属于哪类流动。接着，作者分别讨论了滑流的应力和边界条件、小 M 数滑流的边界条件、大 M 数自由分子流以及流过倾斜平板的自由分子流及其作用下的升力和阻力系数。作者这里所提出的关于流动区域的划分被人们认为是研究稀薄气体力学的开创性工作。

Rough draft, "Double spacing"
"One carbon"

Superaerodynamics, Mechanics of Rarefied Gases

H. S. Tsien

Introduction

A. F. Zahm (Ref. 1) in 1934 published an article on the aerodynamics of highly rarefied gases, a branch of hydrodynamics which he called superaerodynamics. Flight at the extreme altitude with the means of propulsion then available, seemed at that time to be hardly realizable, and the subject must be considered as of academic interest rather than of practical engineering importance. With the recent perfection of rocket as a propulsive power plant, the situation is radically changed and there might seem to be no limit on the altitude that can be reached by an aircraft. There are even indications for the optimum altitude flight of long range rocket airplane to be approximately 60 miles. At such high altitudes, the air density is so low that the fluid must be thought of as being a coarse structure instead of as a continuous medium in the conventional hydrodynamics, and concepts of superaerodynamics are needed to guide the design of such aircraft.

Besides the applications of superaerodynamics to high altitude flights, there are many other applications to industrial processes where low density gases are involved. The knowledge of superaerodynamics could be of invaluable help in improving the efficiency of these processes. For instance, the pumping of low density gases will be of more and more importance due to the use of high vacuum techniques. The improvement in the design of such gas pumps needs definitely the understanding of the principles of mechanics of low density gases. It is the purpose of the present paper

(Here and there the ground was broken by a few physicists) 2

to discuss the fundamental concepts of this new branch of fluid mechanics and indicate some of the results obtained already. Of course, ~~the field of superaerodynamics is still absolutely uncultivated. & However~~ further efforts are definitely required to develop this branch of fluid mechanics into an aid in design engineering and research.

~~Space~~ Gas at Low Pressures

As a first approximation, the gas may be considered as an aggregate of rapidly moving particles which are constantly colliding with each other. The influence of the particles on each other can be conveniently neglected until they are so close together that a "collision" takes place. Then the coarseness of the structure of the gaseous medium can be expressed by the parameter l which is the mean distance the particles or molecules travelled between collisions. Since the instantaneous velocities distribution and density distribution in the gas are far from uniform, one must use the statistical average of a quantity instead of the instantaneous value of the quantity. The distance l is then the statistical average of the billions and billions of molecules concerned. The average l is called the mean free path of the gas. If the mean free path is very small in comparison with the dimension of the flow field or the dimension of the body in the flow field then the gas can be considered as a continuum and the ordinary gasdynamics is sufficient for the analysis of these flows. If the mean free path is not negligible when compared with the dimension of the body, then the effects of the discrete character of the gas must be taken into account in the calculation. Then if L is the linear dimension of the body, then superaerodynamics can be

defined as the aerodynamics of flows where the ratio l/L is not negligible.

The free mean path l can be calculated from the mean velocity of the molecules, the density of the gas and the "effective radius of the molecules". However there is no direct way of measuring the "effective radius of the molecules", therefore it is better to express the quantity l in terms of a measurable quantity such as viscosity of the gas. This is easily done by the following consideration. If the gas flows in the x-direction with macroscopic velocity $u(y)$, the gradient of velocity is then du/dy. Now the gas molecules in a lower layer moves into a upper layer with the average velocity $\bar{v}$ of the molecules. The distance they will travel before they lose their identity by collision with other molecules is l. The difference in macroscopic velocity of the layer where the molecules originated and the layer where the molecules are mixed is then $l\,\frac{du}{dy}$. The mass of molecules crosses the a unit area of the layer is proportional to $\rho\bar{v}$ where ρ is the density of the fluid. Therefore momentum exchange between layers is proportional to $\rho\bar{v}\,l\,\frac{du}{dy}$. This is the viscous shearing stress τ. Since the coefficient of viscosity is defined as

$$\tau = \mu\,\frac{du}{dy}$$

μ is proportional to $\rho\bar{v}l$. The proportionality constant can be calculated by the kinetic theory. The most accurate one is that due to S. Chapman (Ref 2) who gives

$$\mu = 0.499\,\rho\bar{v}l \qquad (1)$$

The average velocity $\bar{v}$ is closely connected with the "velocity of sound" a by the equation

$$\bar{v} = \sqrt{\frac{8}{\pi}}\sqrt{\frac{p}{\rho}} = \sqrt{\frac{8}{\pi\gamma}}\cdot a \qquad (2)$$

where γ is the ratio of specific heats. From Eqs. (1) and (2), the mean free path is given in terms of the kinematic viscosity $\nu = \mu/\rho$ and the velocity of sound a by

$$l = 1.255\sqrt{\gamma}\,\frac{\nu}{a} \qquad (3)$$

For air, the quantity $l\left(\frac{p}{p_0}\right)$ in inches where p and p_0 are the pressure and standard pressure (one atmosphere) respectively, is tabulated in Table 1. Thus for ordinary pressures, the mean free path is truly negligible compared with the body dimensions and therefore gasdynamics is suffice for aerodynamic calculations. However at very high altitude where the pressure may be only one millionth of the pressure at the surface of the earth, the mean free path will be comparable to the body dimensions. A clear demonstration of this fact is perhaps that of Fig. 2, where the mean free path calculated by H. B. Maris (Ref. 3) is plotted against the altitude. At 150 miles altitude, the mean free path is nearly 1 inch. The air there is certainly not a continuum.

The gas molecules have generally three kinds of intrinsic energies: the translational energy, the rotational energy and the vibrational energy. When the gas is at an equilibrium, the distribution of the total energy among the three forms is uniquely determined by the properties of the gas molecules concerned. If this equilibrium is suddenly destroyed by a sudden change in the external conditions such as an expansion, new equilibrium will be reached by numerous collisions of the molecules under these new conditions. However it is known through ultrasonic dispersion etc., that the process to reach the equilibrium of the vibrational energy is a very slow one. In other words, a very large number of collisions are necessary to excite the vibration

degrees of freedom of the molecules properly. Since the average velocity of the molecules is $\bar{v}$, the average distance travelled by the molecules per unit time is $\bar{v}\cdot 1$. The number of collision made by the molecules per unit time is then $\bar{v}/l$. Therefore the time τ, generally called the relaxation time, necessary to excite the vibrational energy is then proportional to the mean free path at a given temperature. In other words, the relaxation time is inversely proportional to the pressure of the gas. The measured value of τ for several typical gases is given in Table 2. (Ref. 4) Therefore at very low pressures, the relaxation time may be of the order of one second. If this is coupled with high flow velocity, the gas must have passed the body before the vibrational degrees of freedom are excited. Therefore low density gas flow, the vibrational energy of the gas can be considered as fixed at the "free stream" value. This decrease in the number of degrees of freedom of the gas molecules tends to raise the value of γ, the ratio of specific heats. It is fortunate that for air at ordinary temperatures, the equilibrium vibrational energy is quite small, therefore this freezing of vibrational degrees will not greatly alter the value of γ. However for other polyatomic gases, the effect may be appreciable.

3. The Parameter l/L and Different Regimes of Fluid Mechanics

If the mean free path l is very small compared with the dimension L of the body, say the length L, then the fluid can be considered as a continuum and the calculation can be made by using the methods of gasdynamics. When the mean free path l is small but not negligible when compared with the body dimension L, the flow conditions near the wall were first found out by J. C

Maxwell in 1879. R. A. Millikan (Ref. 5) has reduced Maxwell's consideration to a very simple form. A more detailed theoretical treatment of the problem by Boltzmann's H-theorem was given by P. S. Epstein (Ref. 6). The results of these investigations show that at the wall the flow velocity u and the normal gradient $\partial u/\partial y$ of u are connected by the relation

$$\beta u = \mu \frac{\partial u}{\partial y}, \quad y=0 \qquad (6)$$

where β is the coefficient of external sliding friction and μ the coefficient of viscosity. The ratio of μ and β is given by

$$\frac{\mu}{\beta} = 0.998\left(\frac{2}{f} - 1\right) l \qquad (7)$$

f is the fraction of tangential momentum of the impinging molecules transmitted to the wall. Maxwell himself interpreted the fractional value of f as meaning that a fraction f of the surface reflects diffusely and the remainder specularly.

Insert here! → It is seen from Eq. (7) that if l is very small compared with the scale of the velocity gradient or the thickness of the boundary layer then β is very large and u at the wall is very small. This is the reason why in aerodynamics one generally imposes the boundary condition u=0 at the wall. The values of f were given by Millikan (Ref. 5) as listed in Table 3.

If l/δ is not negligible then there is a slip at the wall.

Similar to the velocity discontinuity, there is a temperature discontinuity at the wall first found by M. v. Smoluchowski in 1898. Then

$$\alpha(T - T_w) = \lambda \frac{\partial T}{\partial y}, \quad y=0 \qquad (8)$$

where T is temperature of the gas, T_w is the temperature of the wall and α is the coefficient of temperature discontinuity. This coefficient is

Insert to p. 6

¶

Table 3 shows that the value of f is very nearly unity. Therefore the molecules after striking the surface have a tendency (strong) to reflect diffusely. On the scale of molecular dimension the surface of a solid body must be extremely rugged even if it is highly polished in the macroscopic sense. Then we can not expect the molecules to reflect in any uniform direction & the tendency towards diffuse reflection is thus, perhaps, expected. Furthermore a closer study of the behavior of the molecules colliding with the solid surface seems to indicate the adsorption (temporary) of the molecules on the surface. The combined structure of gas molecule and the surface materials then breaks up with a certain average rate determined by the "life time" of the combined structure. The molecules thus freed are re-emitted from the surface. With this procedure the molecules re-emerge from the surface with all its history (previous) obliterated. In particular, the direction of re-emission is completely independent upon the direction of impinging. In other words, the molecules re-emit diffusely.

The adaptation of the impinging molecules to the wall conditions is, of course, expected from the temporary adsorption of molecules on the surface, as mentioned before.

can be expressed in terms of the mean free path by a consideration very similar to Maxwell's (Ref. 7). The result is

$$\frac{1}{\beta} = 1.996\left(\frac{2-\alpha}{\alpha}\right)\frac{\gamma}{\gamma+1}\frac{1}{\mu c_p}\, l \qquad (3)$$

where α is accommodation coefficient introduced by M. Knudsen (Ref. 8). It can be defined as the fractional extent to which those molecules that fall on the surface and are reflected or re-emitted from it, have their mean energy adjusted or "accommodated" toward what it would be if the returning molecules were issuing as a stream out of mass of gas at the temperature of the wall. Thus if E_i is the energy brought up to unit area of the wall per second by the incident molecules and E_r the energy carried away by the re-emitted molecules and E_w the energy that would be carried away if the gas is at the wall temperature, then

$$\bar{E}_i - \bar{E}_r = \alpha(\bar{E}_i - E_w) \qquad (4)$$

Of course, strictly speaking, α might be different for the three types of molecular energies: translational, rotational and vibrational. However, experimental evidence seem to indicate that no difference is noticed. The older measured values of α are given in many books on Kinetic theory of gases (Ref. 9). M. L. Wiedmann (Ref. 10) recently determined the value of α for air on metals. The results are given Table 4. Wiedmann concluded that the value of the accommodation coefficient is [illegible] affected by the nature of the technical surfaces of the metal. This again suggest the [illegible] heterogeneous character of the surfaces [illegible].

Summarizing, we can say that if the mean free path is a fraction of the boundary layer thickness δ, the boundary condition at the wall must be modified according to Eqs. (2) and (3). However since in Eq. (3)

$$\frac{l}{\delta} = \frac{l}{L}\frac{L}{\delta} \sim \frac{L}{\delta}\frac{\gamma}{R}\frac{1}{L_0} \sim \frac{L}{\delta}\frac{M}{Re} \qquad (5)$$

where M is the Mach number of the free stream and Re is the Reynolds' number of the flow referred to the length L of the body. For very small Reynolds' number, $L/\delta \sim 1$, then

$$\frac{l}{\delta} \sim \frac{M}{Re}, \qquad Re \ll 1 \qquad (10)$$

For very large Reynolds' number, it is well known that $\frac{L}{\delta} \sim \sqrt{Re}$. Then

$$\frac{l}{\delta} \sim \frac{M}{\sqrt{Re}} \qquad Re \gg 1 \qquad (11)$$

Therefore if the interval $\frac{1}{100} < \frac{l}{\delta} < 1$ is considered as the proper range for the slip-flow, then in the plane of M and Re, this realm of fluid mechanics occupies a region as shown in Fig. 3. The region below this slip-flow region belongs to the realm of the conventional gasdynamics with the usual boundary conditions at the wall.

If the mean free path is much larger compared with the body dimensions one enters an entirely new realm of fluid mechanics. Here the chances for the collisions of molecules among themselves are much smaller than the chances for the collision of molecules with the wall or the surface of the body. Therefore for the calculation of forces, one need only consider the impact of a stream of molecules with a velocity and energy distribution determined by the thermal equilibrium, the Maxwellian distribution. The re-emission of the molecules from the surface will be governed by the accommodation coefficient. But the greatest simplification comes from the fact that one need not consider the distortion of the Maxwellian distribution due to the collision of the re-emitted molecules with the molecules in the stream. The realm of fluid mechanics can be thus called the free molecule flow. Such flows in a tube was first studied by Knudsen (Ref. 10). Epstein (Ref. 11) calculated the drag of a sphere

when the velocity is small.

If one takes the limiting ratio of l/L to be 10 for the free molecule flow, then this realm of fluid mechanics occupies a region of the M-Re plane given by

$$\frac{l}{L} \sim \frac{M}{Re} > 10$$

This is shown in Fig. 3. The region between the free molecule flow and the slip flow is realm of fluid mechanics where the collision between molecules and the collision of molecules with the wall are of equal importance. The problem is very complicated and no satisfactory theoretical solution can yet be offered. However it is certain that as far as the fluid itself is concerned, the characteristic parameters are still the Mach number M and the Reynolds' number Re as used in gas dynamics. The new elements are two: First, two new parameters f and α, specifying the interaction between the gas and the wall are introduced. Secondly, for a given Mach number, the boundary conditions at the surface themselves depend upon the Reynolds' number, while for gas dynamics flows the boundary conditions at the surface do not depend upon the Reynolds' number. This fact makes the drag formula much more complicated as will be evident in the following discussions.

Slip Flow at Small Mach Numbers

If the Mach number is very small, the effect of the compressibility of the gas can be neglected and the fluid can be treated as incompressible. In order that the mean free path will be comparable with body dimension, the Reynolds' number must be also very small. The flow can then be calculated by the well-known method of Stokes (Ref. 13) who neglected the inertia terms, being small compared with the viscous and pressure

1

Free Molecule Flow with Dissociation Over an Inclined Flat Plate

Ref: J. R. Stalder, D. Jukoff: "Heat Transfer to Bodies Travelling at High Speed in the Upper Atmosphere" NACA TN 1682 (1948)

J. R. Stalder, G. Goodwin, M. O. Creager: "A Comparison of Theory and Experiment for High Speed Free-Molecule Flow" NACA TN 2244 (1950)

1) Impinging quantities

s = molecular speed ratio = $\frac{U}{c_i}$

$$m_i = \rho^0 \frac{c_i}{2\sqrt{\pi}} \left\{ e^{-s^2 \sin^2\theta} + \sqrt{\pi}\, s \sin\theta \left[1 + \operatorname{erf}(s \sin\theta) \right] \right\}$$

$$p_i = \frac{1}{2} \rho^0 U^2 \left\{ \frac{1}{\sqrt{\pi}} \frac{\sin\theta}{s} e^{-s^2 \sin^2\theta} + \left[\frac{1}{2s^2} + \sin^2\theta \right] \left[1 + \operatorname{erf}(s \sin\theta) \right] \right\}$$

$$\tau_i = \frac{1}{2} \rho^0 U^2 \left\{ \frac{1}{\sqrt{\pi}} \frac{\cos\theta}{s} e^{-s^2 \sin^2\theta} + \sin\theta \cos\theta \left[1 + \operatorname{erf}(s \sin\theta) \right] \right\}$$

$$E_i = \rho^0 \frac{c_i}{2\sqrt{\pi}} \left\{ \left[h^0 + \frac{1}{2} U^2 - \frac{1}{2} R' T^0 \right] e^{-s^2 \sin^2\theta} + \sqrt{\pi} \left[\frac{1}{2} U^2 + h^0 \right] s \sin\theta \left[1 + \operatorname{erf}(s \sin\theta) \right] \right\}$$

2) Re-emission

$$E_r = m_i \left(h_1 - \frac{1}{2} R' T_w \right) + \frac{K_p(T_w)}{4\sqrt{2\pi R T_w}} \left\{ -1 + \sqrt{1 + \frac{8 m_i \sqrt{2\pi R T_w}}{K_p(T_w)}} \right\} \left[\Delta h - \frac{1}{2} R T_w \right]$$

2

$$p_T = \sqrt{\frac{\pi}{2} R T_w}\left[m_i + (\sqrt{2}-1)\frac{K_p(T_w)}{4\sqrt{2\pi R T_w}}\left\{-1+\sqrt{1+\frac{\ell m_i \sqrt{2\pi R T_w}}{K_p(T_w)}}\right\}\right]$$

III) Inclined plate at α

the energy balance,

$$m_{i+} = \rho^0 \frac{c_i}{2\sqrt{\pi}}\left\{ e^{-s^2 \sin^2\alpha} + \sqrt{\pi}\, s \sin\alpha \left[1+\mathrm{erf}(s \sin\alpha)\right]\right\}$$

$$m_{i-} = \rho^0 \frac{c_i}{2\sqrt{\pi}}\left\{ e^{-s^2 \sin^2\alpha} - \sqrt{\pi}\, s \sin\alpha \left[1-\mathrm{erf}(s \sin\alpha)\right]\right\}$$

$$E_{i+} = \rho^0 \frac{c_i}{2\sqrt{\pi}}\left\{ \left[h^0+\frac{1}{2}U^2-\frac{1}{2}RT^0\right] e^{-s^2 \sin^2\alpha} + \sqrt{\pi}\left[\frac{1}{2}U^2+h^0\right] s \sin\alpha \left[1+\mathrm{erf}(s \sin\alpha)\right]\right\}$$

$$E_{i-} = \rho^0 \frac{c_i}{2\sqrt{\pi}}\left\{ \left[h^0+\frac{1}{2}U^2-\frac{1}{2}RT^0\right] e^{-s^2 \sin^2\alpha} - \sqrt{\pi}\left[\frac{1}{2}U^2+h^0\right] s \sin\alpha \left[1-\mathrm{erf}(s \sin\alpha)\right]\right\}$$

$$E_{i+} + E_{i-} = \rho^0 \frac{c_i}{\sqrt{\pi}}\left\{ \left[h^0+\frac{1}{2}U^2-\frac{1}{2}RT^0\right] e^{-s^2 \sin^2\alpha} + \sqrt{\pi}\left[\frac{1}{2}U^2+h^0\right] s \sin\alpha\, \mathrm{erf}(s \sin\alpha)\right\}$$

$$E_{r+} = m_{i+}\left(h_1-\frac{1}{2}R^{\prime}T_w\right) + \frac{K_p(T_w)}{4\sqrt{2\pi R^{\prime} T_w}}\left\{-1+\sqrt{1+\frac{\ell m_{i+}\sqrt{2\pi R T_w}}{K_p(T_w)}}\right\}\left[\Delta h-\frac{1}{2}RT_w\right]$$

$$E_{r-} = m_{i-}\left(h_1-\frac{1}{2}R^{\prime}T_w\right) + \frac{K_p(T_w)}{4\sqrt{2\pi R^{\prime} T_w}}\left\{-1+\sqrt{1+\frac{\ell m_{i-}\sqrt{2\pi R T_w}}{K_p(T_w)}}\right\}\left[\Delta h-\frac{1}{2}RT_w\right]$$

So

$$E_{i+} + E_{i-} = E_{r+} + E_{r-} + \varepsilon\sigma\left(T_w^4 - T^{0\,4}\right)$$

2

$$\left[h^0+\tfrac{1}{2}u^2-\tfrac{1}{2}RT^0\right]e^{-s^2\sin^2\alpha}+\sqrt{\pi}\left[\tfrac{1}{2}u^2+h^0\right]s\sin\alpha\,\operatorname{erf}(s\sin\alpha)$$

$$=\left(h_1-\tfrac{1}{2}RT_w\right)\left[e^{-s^2\sin^2\alpha}+\sqrt{\pi}\,s\sin\alpha\,\operatorname{erf}(s\sin\alpha)\right]+\varepsilon\sigma\left(T_w^4-T^{0\,4}\right)\frac{\sqrt{\pi}}{\rho^0\bar{c}_2}$$

$$+\left(\Delta h-\tfrac{1}{2}RT_w\right)\frac{K_p(T_w)}{4c_{2i}\sqrt{2\pi RT_w}\,\rho^0}\left\{-2+\sqrt{1+\frac{8m_{i+}\sqrt{2\pi RT_w}}{K_p(T_w)}}+\sqrt{1+\frac{8m_{i-}\sqrt{2\pi RT_w}}{K_p(T_w)}}\right\}$$

$$\frac{8m_{i+}\sqrt{2\pi RT_w}}{K_p(T_w)}=\frac{4\rho^0\bar{c}_2^2}{K_p(T_w)}\sqrt{\frac{T_w}{T^0}}\left\{e^{-s^2\sin^2\alpha}+\sqrt{\pi}\,s\sin\alpha\left[1+\operatorname{erf}(s\sin\alpha)\right]\right\}$$

$$=\frac{8\rho^0}{K_p(T_w)}\sqrt{\frac{T_w}{T^0}}\left\{e^{-s^2\sin^2\alpha}+\sqrt{\pi}\,s\sin\alpha\left[1+\operatorname{erf}(s\sin\alpha)\right]\right\}$$

$$\frac{8m_{i-}\sqrt{2\pi RT_w}}{K_p(T_w)}=\frac{8\rho^0}{K_p(T_w)}\sqrt{\frac{T_w}{T^0}}\left\{e^{-s^2\sin^2\alpha}-\sqrt{\pi}\,s\sin\alpha\left[1-\operatorname{erf}(s\sin\alpha)\right]\right\}$$

$\therefore$

$$\left[h^0+RT^0\left(s^2-\tfrac{1}{2}\right)\right]e^{-s^2\sin^2\alpha}+\sqrt{\pi}\left[RT^0s^2+h^0\right]s\sin\alpha\,\operatorname{erf}(s\sin\alpha)$$

$$=\left(h_1-\tfrac{1}{2}RT_w\right)\left[e^{-s^2\sin^2\alpha}+\sqrt{\pi}\,s\sin\alpha\,\operatorname{erf}(s\sin\alpha)\right]+\sqrt{\pi}\,\varepsilon\sigma\left(T_w^4-T^{0\,4}\right)/\rho^0\bar{c}_2$$

$$+\left(\Delta h-\tfrac{1}{2}RT_w\right)\frac{K_p(T_w)}{8\rho^0}\sqrt{\frac{T^0}{T_w}}\left[-2+\sqrt{1+\frac{8\rho^0}{K_p(T_w)}\sqrt{\frac{T_w}{T^0}}\left\{e^{-s^2\sin^2\alpha}+\sqrt{\pi}\,s\sin\alpha\left[1+\operatorname{erf}(s\sin\alpha)\right]\right\}}\right.$$

$$\left.+\sqrt{1+\frac{8\rho^0}{K_p(T_w)}\sqrt{\frac{T_w}{T^0}}\left\{e^{-s^2\sin^2\alpha}-\sqrt{\pi}\,s\sin\alpha\left[1-\operatorname{erf}(s\sin\alpha)\right]\right\}}\right]$$

4

$$c_n = \frac{2}{\sqrt{\pi}} \frac{\sin\alpha}{s} e^{-s^2 \sin^2\alpha} + 2\left[\frac{1}{2s^2} + \sin^2\alpha\right] \operatorname{erf}(s\sin\alpha)$$

$$+ \frac{1}{s^2}\sqrt{\frac{T_w}{T^{\infty}}}\left[\sqrt{\pi}\, s\sin\alpha + (\sqrt{2}-1)\frac{K_p(T_w)}{\ell p^{\infty}}\sqrt{\frac{T^{\infty}}{T_w}}\left\{\sqrt{1+\frac{\ell p^{\infty}}{K_p(T_w)}\sqrt{\frac{T_w}{T^{\infty}}}\left\{e^{-s^2\sin^2\alpha} + \sqrt{\pi}\, s\sin\alpha\left[1+\operatorname{erf}(s\sin\alpha)\right]\right\}}\right.\right.$$

$$\left.\left. - \sqrt{1+\frac{\ell p^{\infty}}{K_p(T_w)}\sqrt{\frac{T_w}{T^{\infty}}}\left\{e^{-s^2\sin^2\alpha} - \sqrt{\pi}\, s\sin\alpha\left[1-\operatorname{erf}(s\sin\alpha)\right]\right\}}\right\}\right]$$

$$c_t = \frac{2}{\sqrt{\pi}} \frac{\cos\alpha}{s} e^{-s^2\sin^2\alpha} + 2\sin\alpha\cos\alpha \operatorname{erf}(s\sin\alpha)$$

$$\frac{1}{2}u^2 - \frac{1}{2}RT^{\infty} = RT^{\infty}\left[\frac{u^2}{2RT^{\infty}} - \frac{1}{2}\right] = RT^{\infty}\left[s^2 - \frac{1}{2}\right]$$

$$\frac{\sqrt{\pi}\,\varepsilon\sigma(T_w^4 - T_{\infty}^4)}{\rho^{\infty} c_i} = \frac{\sqrt{\pi}\,\varepsilon\sigma T_{\infty}^4\left(\frac{T_w^4}{T^{\infty 4}} - 1\right)}{\rho^{\infty} c_i RT^{\infty}} RT^{\infty}$$

$$= \frac{\sqrt{\pi}\,\varepsilon\sigma T_{\infty}^4\left(\frac{T_w^4}{T^{\infty 4}} - 1\right)}{\rho^{\infty} c_i} RT^{\infty}$$

1.1.4
高超声速和跨声速流动的相似律

1.1.4.1
Similarity Laws of Hypersonic Flows
高超声速流动的相似律

这是发表于 1946 年的 “Similarity Laws of Hypersonic Flows ”（高超声速流动的相似律）一文的原始推导手稿，共有 12 页。

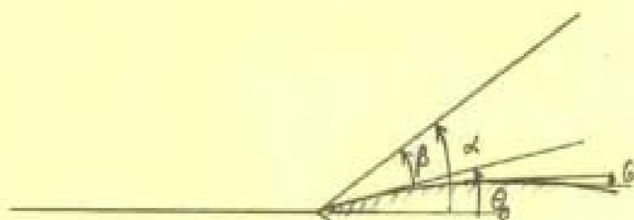
在 40 年代初，对于做超声速飞行的尖头细长物体（如火箭）来说，飞行体周围的流场一般可以采用线性化近似方法来求解。然而，如果流动马赫数很高，流场中出现强击波，流动不再具有位势，线性化方程不再适用。在这种情况下，一方面，人们正在为研究这种高超声速流动而设计建造高超声速风洞；另一方面，也积极探讨新的理论途径。

那时，作者正在帮助 Theodore von Kármán（冯・卡门）整理发表有关跨声速流动相似律的论文，von Kármán 用了一个仿射变换，建立了跨声速流动的相似律。作者意识到，这种方法同样可以用来分析高超声速流动的控制方程。一般来说，描述这种流动的独立的无量纲参数有两个，即飞行体的相对厚度 δ/b（δ 和 b 分别是飞行体的厚度和长度）以及来流马赫数 M 。作者在分析了控制方程中各项大小以后发现：在飞行体比较细长的情况下，可以舍弃一些高阶小量，由此得到的简化近似方程只包含一个无量纲参数，它就是上述两个参数的乘积 K（$= M\delta/b$）。由此，作者就得到了有关高超声速流动的升力和阻力系数的相似律。有了这一相似律，可以大大减少风洞试验和数值计算的工作量。

Hypersonic Flow $M_1 \to \infty$

(I) Two Dimensional Case

a) The initial shock



$$\boxed{\tan\beta = \tan\alpha \cdot \frac{\gamma-1}{\gamma+1}}$$

$$\boxed{p_2 = \frac{2}{\gamma+1}\rho_1 v_1^2 \sin^2\alpha}$$

$$M_2^2 = (1+\cot^2\beta)\frac{\gamma-1}{2\gamma} = \left\{1+\cot^2\alpha\cdot\left(\frac{\gamma+1}{\gamma-1}\right)^2\right\}\frac{\gamma-1}{2\gamma}$$

$$M_2^2 = \frac{\gamma-1}{2\gamma}\frac{1}{\sin^2\alpha} + \frac{2}{\gamma-1}\cot^2\alpha$$

$$= \frac{\gamma-1}{2\gamma}\frac{1}{\sin^2\alpha} + \frac{2}{\gamma-1}\left[\frac{1}{\sin^2\alpha} - 1\right]$$

$$\boxed{M_2^2 = \frac{(\gamma+1)^2}{2\gamma(\gamma-1)}\frac{1}{\sin^2\alpha} - \frac{2}{\gamma-1}}$$

$$\cot\beta = \cot\alpha\left(\frac{\gamma+1}{\gamma-1}\right)$$

2

b) Subsequent Expansion

We shall use the formula

$$\theta_0 - \theta = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1}\sqrt{\frac{M^2-1}{\frac{\gamma+1}{\gamma-1}}} - \tan^{-1}\sqrt{M^2-1} - \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1}\sqrt{\frac{M_2^2-1}{\frac{\gamma+1}{\gamma-1}}} - \tan^{-1}\sqrt{M_2^2-1}$$

$$= \sqrt{\frac{\gamma+1}{\gamma-1}}\left[\frac{\pi}{2} - \sqrt{\frac{\gamma+1}{\gamma-1}}\frac{1}{\sqrt{M^2-1}} + \frac{1}{3}\left(\frac{\gamma+1}{\gamma-1}\right)^{3/2}\frac{1}{(M^2-1)^{3/2}} \cdots\right]$$

$$- \left[\frac{\pi}{2} - \frac{1}{\sqrt{M^2-1}} + \frac{1}{3}\frac{1}{(M^2-1)^{3/2}} \cdots\right]$$

$$- \sqrt{\frac{\gamma+1}{\gamma-1}}\left[\frac{\pi}{2} - \sqrt{\frac{\gamma+1}{\gamma-1}}\frac{1}{\sqrt{M_2^2-1}} + \frac{1}{3}\left(\frac{\gamma+1}{\gamma-1}\right)^{3/2}\frac{1}{(M_2^2-1)^{3/2}} \cdots\right]$$

$$+ \left[\frac{\pi}{2} - \frac{1}{\sqrt{M_2^2-1}} + \frac{1}{3}\frac{1}{(M_2^2-1)^{3/2}} \cdots\right]$$

$$= \frac{2}{\gamma-1}\left(\frac{1}{\sqrt{M_2^2-1}} - \frac{1}{\sqrt{M^2-1}}\right) - \frac{1}{3}\frac{4\gamma}{(\gamma-1)^2}\left(\frac{1}{(M_2^2-1)^{3/2}} - \frac{1}{(M^2-1)^{3/2}}\right) \cdots$$

We have the formula

$$1 + \frac{\gamma-1}{2}M^2 = \left(\frac{p_0'}{p}\right)^{\frac{\gamma-1}{\gamma}}, \qquad 1 + \frac{\gamma-1}{2}M_2^2 = \left(\frac{p_0'}{p_2}\right)^{\frac{\gamma-1}{\gamma}}$$

Our aim is to express p/p_2 in terms of $\theta_0 - \theta$.

$$\boxed{\frac{1+\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}M_2^2} = \left(\frac{p_2}{p}\right)^{\frac{\gamma-1}{\gamma}}}$$

2

$$\theta_2-\theta=\frac{2}{\gamma-1}\frac{1}{\sqrt{M_2^2-1}}\left\{1-\sqrt{\frac{M_2^2-1}{M^2-1}}\right\}-\frac{1}{3}\frac{4\gamma}{(\gamma-1)^2}\frac{1}{(M_2^2-1)^{3/2}}\left\{1-\left(\frac{M_2^2-1}{M^2-1}\right)^{3/2}\right\}\cdots$$

$$=\frac{2}{\gamma-1}\frac{1}{(M_2^2-1)^{1/2}}\left\{1-\left(\frac{1-\frac{1}{M_2^2}}{\left(\frac{M}{M_2}\right)^2-\frac{1}{M_2^2}}\right)^{1/2}\right\}-\frac{1}{3}\frac{4\gamma}{(\gamma-1)^2}\frac{1}{(M_2^2-1)^{3/2}}\left\{1-\left(\frac{1-\frac{1}{M_2^2}}{\left(\frac{M}{M_2}\right)^2-\frac{1}{M_2^2}}\right)^{3/2}\right\}\cdots$$

$$\frac{\frac{1}{M_2^2}+\frac{\gamma-1}{2}\left(\frac{M}{M_2}\right)^2}{\frac{1}{M_2^2}+\frac{\gamma-1}{2}}=\left(\frac{p_2}{p}\right)^{\frac{\gamma-1}{\gamma}}$$

$$\frac{1}{M_2^2}+\frac{\gamma-1}{2}\left(\frac{M}{M_2}\right)^2=\left(\frac{1}{M_2^2}+\frac{\gamma-1}{2}\right)\left(\frac{p_2}{p}\right)^{\frac{\gamma-1}{\gamma}},\quad \left(\frac{M}{M_2}\right)^2=\frac{2}{\gamma-1}\left\{\left(\frac{1}{M_2^2}+\frac{\gamma-1}{2}\right)\left(\frac{p_2}{p}\right)^{\frac{\gamma-1}{\gamma}}-\frac{1}{M_2^2}\right\}$$

$$\left(\frac{M}{M_2}\right)^2-\frac{1}{M_2^2}=\left\{\frac{2}{(\gamma-1)M_2^2}+1\right\}\left(\frac{p_2}{p}\right)^{\frac{\gamma-1}{\gamma}}-\left(\frac{\gamma+1}{\gamma-1}\right)\frac{1}{M_2^2}$$

$$=\left(\frac{p_2}{p}\right)^{\frac{\gamma-1}{\gamma}}+\frac{1}{\gamma-1}\left\{2\left(\frac{p_2}{p}\right)^{\frac{\gamma-1}{\gamma}}-(\gamma+1)\right\}\frac{1}{M_2^2}$$

$$=\left(\frac{p_2}{p}\right)^{\frac{\gamma-1}{\gamma}}\left[1+\frac{1}{\gamma-1}\left\{2-(\gamma+1)\left(\frac{p}{p_2}\right)^{\frac{\gamma-1}{\gamma}}\right\}\frac{1}{M_2^2}\right]$$

$$\frac{1-\frac{1}{M_2^2}}{\left(\frac{M}{M_2}\right)^2-\frac{1}{M_2^2}}=\left(\frac{p}{p_2}\right)^{\frac{\gamma-1}{\gamma}}\left[1-\left\{1+\frac{2}{\gamma-1}-\frac{\gamma+1}{\gamma-1}\left(\frac{p}{p_2}\right)^{\frac{\gamma-1}{\gamma}}\right\}\frac{1}{M_2^2}\cdots\right]$$

$$=\left(\frac{p}{p_2}\right)^{\frac{\gamma-1}{\gamma}}\left[1-\frac{\gamma+1}{\gamma-1}\left\{1-\left(\frac{p}{p_2}\right)^{\frac{\gamma-1}{\gamma}}\right\}\frac{1}{M_2^2}\cdots\right]$$

4

$$\theta_0-\theta=\frac{2}{\gamma-1}\frac{1}{M_2}\left\{1+\frac{1}{2}\frac{1}{M_2^2}\cdots\right\}\left[1-\left(\frac{p}{p_2}\right)^{\frac{\gamma-1}{2\gamma}}\left\{1-\frac{1}{2}\frac{\gamma+1}{\gamma-1}\left(1-\left(\frac{p}{p_2}\right)^{\frac{\gamma-1}{\gamma}}\right)\frac{1}{M_2^2}\cdots\right\}\right]$$

$$-\frac{1}{3}\frac{4\gamma}{(\gamma-1)^2}\frac{1}{M_2^2}\left\{1+\frac{3}{2}\frac{1}{M_2^2}\cdots\right\}\left[1-\left(\frac{p}{p_2}\right)^{\frac{3(\gamma-1)}{2\gamma}}\left\{1-\frac{3}{2}\frac{\gamma+1}{\gamma-1}\left(1-\left(\frac{p}{p_2}\right)^{\frac{\gamma-1}{\gamma}}\right)\frac{1}{M_2^2}\cdots\right\}\right]$$

$$=\frac{2}{\gamma-1}\frac{1}{M_2}\Bigg[\left\{1+\frac{1}{2}\frac{1}{M_2^2}\cdots\right\}\left\{\left(1-\left(\frac{p}{p_2}\right)^{\frac{\gamma-1}{2\gamma}}\right)+\frac{1}{2}\frac{\gamma+1}{\gamma-1}\left(\frac{p}{p_2}\right)^{\frac{\gamma-1}{2\gamma}}\left(1-\left(\frac{p}{p_2}\right)^{\frac{\gamma-1}{\gamma}}\right)\frac{1}{M_2^2}\cdots\right\}$$

$$-\frac{1}{3}\frac{2\gamma}{\gamma-1}\frac{1}{M_2^2}\left\{1-\left(\frac{p}{p_2}\right)^{\frac{3(\gamma-1)}{2\gamma}}\right\}\cdots\Bigg]$$

$$=\frac{2}{\gamma-1}\frac{1}{M_2}\Bigg[\left\{1-\left(\frac{p}{p_2}\right)^{\frac{\gamma-1}{2\gamma}}\right\}+\frac{1}{2}\Bigg\{\left[1-\left(\frac{p}{p_2}\right)^{\frac{\gamma-1}{2\gamma}}\right]+\frac{\gamma+1}{\gamma-1}\left(\frac{p}{p_2}\right)^{\frac{\gamma-1}{2\gamma}}\left[1-\left(\frac{p}{p_2}\right)^{\frac{\gamma-1}{\gamma}}\right]$$

$$-\frac{1}{3}\frac{4\gamma}{\gamma-1}\left[1-\left(\frac{p}{p_2}\right)^{\frac{3(\gamma-1)}{2\gamma}}\right]\Bigg\}\frac{1}{M_2^2}\cdots\Bigg]$$

Thus

$$\boxed{\theta_0-\theta=\frac{2}{\gamma-1}\frac{1}{M_2}\left[\left\{1-\left(\frac{p}{p_2}\right)^{\frac{\gamma-1}{2\gamma}}\right\}+\frac{1}{2}\left\{-\frac{\gamma+3}{3(\gamma-1)}+\frac{2}{\gamma-1}\left(\frac{p}{p_2}\right)^{\frac{\gamma-1}{2\gamma}}-\frac{3-\gamma}{3(\gamma-1)}\left(\frac{p}{p_2}\right)^{\frac{3(\gamma-1)}{2\gamma}}\right\}\frac{1}{M_2^2}\cdots\right]}$$

(c) Small θ_0

$$\tan(\alpha-\theta_0)=\frac{\gamma-1}{\gamma+1}\tan\alpha$$

$$\frac{\tan\alpha-\tan\theta_0}{1+\tan\alpha\cdot\tan\theta_0}=\frac{\gamma-1}{\gamma+1}\tan\alpha$$

$$\tan\alpha-\tan\theta_0=\frac{\gamma-1}{\gamma+1}\tan\alpha+\frac{\gamma-1}{\gamma+1}\tan^2\alpha\cdot\tan\theta_0$$

$$\frac{2}{\gamma+1}\tan\alpha=\tan\theta_0+\frac{\gamma-1}{\gamma+1}\tan^2\alpha\cdot\tan\theta_0$$

$$\frac{4}{(\gamma+1)^2}\frac{\sin^2\alpha}{1-\sin^2\alpha}=\tan^2\theta_0+2\frac{\gamma-1}{\gamma+1}\tan^2\theta_0\frac{\sin^2\alpha}{1-\sin^2\alpha}+\left(\frac{\gamma-1}{\gamma+1}\right)^2\tan^2\theta_0\frac{\sin^4\alpha}{(1-\sin^2\alpha)^2}$$

5

$$\frac{4}{(\gamma+1)^2}\left[\sin^2\alpha-\sin^4\alpha\right]=\tan^2\theta_0\left[1-2\sin^2\alpha+\sin^4\alpha\right]+2\frac{\gamma-1}{\gamma+1}\tan^2\theta_0\left[\sin^2\alpha-\sin^4\alpha\right]$$

$$+\left(\frac{\gamma-1}{\gamma+1}\right)^2\tan^2\theta_0\,\sin^4\alpha$$

$$\left[\left(1-\frac{\gamma-1}{\gamma+1}\right)^2\tan^2\theta_0+\frac{4}{(\gamma+1)^2}\right]\sin^4\alpha+\left[\left\{2\frac{\gamma-1}{\gamma+1}-2\right\}\tan^2\theta_0-\frac{4}{(\gamma+1)^2}\right]\sin^2\alpha$$

$$+\tan^2\theta_0=0$$

$$\left(\frac{2}{\gamma+1}\right)^2(\tan^2\theta_0+1)\sin^4\alpha-2\frac{2}{\gamma+1}\left[\tan^2\theta_0+\frac{1}{\gamma+1}\right]\sin^2\alpha+\tan^2\theta_0=0$$

$$(\tan^2\theta_0+1)\sin^4\alpha-2\left(\frac{\gamma+1}{2}\right)\left[\tan^2\theta_0+\frac{1}{\gamma+1}\right]\sin^2\alpha+\left(\frac{\gamma+1}{2}\right)^2\tan^2\theta_0=0$$

$$\sin^2\alpha=\cos^2\theta_0\left[\left(\frac{\gamma+1}{2}\right)\left(\tan^2\theta_0+\frac{1}{\gamma+1}\right)-\sqrt{\left(\frac{\gamma+1}{2}\right)^2\tan^4\theta_0+\left(\frac{\gamma+1}{2}\right)\tan^2\theta_0+\frac{1}{4}-\left(\frac{\gamma+1}{2}\right)^2\tan^4\theta_0-\left(\frac{\gamma+1}{2}\right)^2\tan^2\theta_0}\right]$$

$$=\cos^2\theta_0\left[\frac{\gamma+1}{2}\tan^2\theta_0+\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{\gamma^2-1}{4}\tan^2\theta_0}\right]$$

$$=\frac{1}{2}\cos^2\theta_0\left[(\gamma+1)\tan^2\theta_0+1-\sqrt{1-(\gamma^2-1)\tan^2\theta_0}\right]$$

$$=\frac{1}{2}\cos^2\theta_0\left[(\gamma+1)\tan^2\theta_0+1-1+\frac{\gamma^2-1}{2}\tan^2\theta_0+\frac{1}{8}(\gamma^2-1)^2\tan^4\theta_0\cdots\right]$$

$$\sin^2\alpha=\left(\frac{\gamma+1}{2}\right)^2\sin^2\theta_0+\frac{1}{16}(\gamma^2-1)^2\sin^2\theta_0\,\tan^2\theta_0\cdots$$

$$\sin\theta_0=\theta_0\left(1-\frac{1}{6}\theta_0^2\cdots\right)$$

$$\sin^2\theta_0=\theta_0^2\left(1-\frac{1}{3}\theta_0^2\cdots\right),$$

6

$$\sin^2\alpha = \left(\frac{\gamma+1}{2}\right)^2 \left[\theta_0^2 \left(1 - \frac{1}{3}\theta_0^2 \cdots\right) + \frac{1}{4}(\gamma-1)^2 \theta_0^4 \cdots \right]$$

$$\boxed{\sin^2\alpha = \left(\frac{\gamma+1}{2}\right)^2 \theta_0^2 \left[1 - \left\{\frac{1}{3} - \left(\frac{\gamma-1}{2}\right)^2\right\} \theta_0^2 \cdots \right]}$$

$$\frac{1}{M_2^2} = \frac{1}{\dfrac{(\gamma+1)^2}{2\gamma(\gamma-1)} \dfrac{1}{\sin^2\alpha} - \dfrac{2}{\gamma-1}}$$

$$= \frac{\sin^2\alpha}{\dfrac{(\gamma+1)^2}{2\gamma(\gamma-1)} - \dfrac{2}{\gamma-1}\sin^2\alpha}$$

$$= \frac{\dfrac{2\gamma(\gamma-1)}{(\gamma+1)^2}\sin^2\alpha}{1 - \dfrac{4\gamma}{(\gamma+1)^2}\sin^2\alpha}$$

$$\frac{1}{M_2^2} = \frac{\gamma(\gamma-1)}{2}\theta_0^2 \left[1 - \left\{\frac{1}{3} - \left(\frac{\gamma-1}{2}\right)^2\right\}\theta_0^2 \cdots\right]\left[1 + \gamma\theta_0^2 \cdots\right]$$

$$\frac{1}{M_2^2} = \frac{\gamma(\gamma-1)}{2}\theta_0^2 \left[1 + \left\{\gamma + \left(\frac{\gamma-1}{2}\right)^2 - \frac{1}{3}\right\}\theta_0^2 \cdots\right]$$

$$\boxed{\frac{1}{M_2} = \sqrt{\frac{\gamma(\gamma-1)}{2}}\,\theta_0 \left[1 + \frac{1}{2}\left\{\gamma + \left(\frac{\gamma-1}{2}\right)^2 - \frac{1}{3}\right\}\theta_0^2 \cdots\right]}$$

Then

$$\theta_0-\theta=\sqrt{\frac{2\gamma}{\gamma-1}}\,\theta_0\left[1+\frac{1}{2}\left\{\gamma+\left(\frac{\gamma-1}{2}\right)^2-\frac{1}{3}\right\}\theta_0^2\cdots\right]$$

$$\left[\left\{1-\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{2\gamma}}\right\}+\frac{\gamma}{4}\left\{-\frac{\gamma+3}{3}+2\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{2\gamma}}-\frac{3-\gamma}{3}\left(\frac{p}{p_0}\right)^{\frac{3(\gamma-1)}{2\gamma}}\right\}\theta_0^2\cdots\right]$$

Hence

$$\boxed{\theta_0-\theta=\sqrt{\frac{2\gamma}{\gamma-1}}\,\theta_0\left[\left\{1-\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{2\gamma}}\right\}+\left\{\frac{1}{2}\left[\gamma+\left(\frac{\gamma-1}{2}\right)^2-\frac{1}{3}\right]\left[1-\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{2\gamma}}\right]+\frac{\gamma}{4}\left[-\frac{\gamma+3}{3}+2\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{2\gamma}}-\frac{3-\gamma}{3}\left(\frac{p}{p_0}\right)^{\frac{3(\gamma-1)}{2\gamma}}\right]\right\}\theta_0^2\cdots\right]}$$

If $p=0$, then

$$-\theta=\sqrt{\frac{2\gamma}{\gamma-1}}\,\theta_0\left[1+\left\{\frac{1}{2}\left[\gamma+\left(\frac{\gamma-1}{2}\right)^2-\frac{1}{3}\right]-\frac{\gamma(\gamma+3)}{12}\right\}\theta_0^2\cdots\right]-\theta_0$$

$$\sim\left(\sqrt{\frac{2\gamma}{\gamma-1}}-1\right)\theta_0 \qquad \text{If } \gamma=1.405,\quad \sqrt{\frac{2\gamma}{\gamma-1}}-1=1.632$$

Two-Dimensional Flow

The general differential equation is

$$\left(1-\frac{u^2}{a^2}\right)\frac{\partial u}{\partial x} - 2\frac{uv}{a^2}\frac{\partial u}{\partial y} + \left(1-\frac{v^2}{a^2}\right)\frac{\partial v}{\partial y} = 0$$

$$u = U + \frac{\partial \varphi}{\partial x}$$

$$v = \frac{\partial \varphi}{\partial y}$$

$$a^2 = a_0^2 - \frac{\gamma-1}{2}\left[U^2 + 2U\frac{\partial \varphi}{\partial x} + \left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2\right]$$

$$a'^2 = a_0^2 - \frac{\gamma-1}{2}U^2, \qquad a_0^2 = a'^2 + \frac{\gamma-1}{2}U^2$$

$$a^2 = a'^2 + \frac{\gamma-1}{2}U^2 - \frac{\gamma-1}{2}\left[U^2 + 2U\frac{\partial \varphi}{\partial x} + \left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2\right]$$

$$= a'^2 - \frac{\gamma-1}{2}\left[2U\frac{\partial \varphi}{\partial x} + \left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2\right]$$

$$\frac{u^2}{a^2} = \frac{U^2 + 2U\frac{\partial \varphi}{\partial x} + \left(\frac{\partial \varphi}{\partial x}\right)^2}{a'^2\left[1 - \frac{\gamma-1}{2}\left\{2M'\frac{1}{a'}\frac{\partial \varphi}{\partial x} + \frac{1}{a'^2}\left(\frac{\partial \varphi}{\partial x}\right)^2 + \frac{1}{a'^2}\left(\frac{\partial \varphi}{\partial y}\right)^2\right\}\right]}$$

$$\cong \frac{M'^2 + 2M'\frac{1}{a'}\frac{\partial \varphi}{\partial x}}{1 - (\gamma-1)M'\frac{1}{a'}\frac{\partial \varphi}{\partial x} - \frac{\gamma-1}{2}\frac{1}{a'^2}\left(\frac{\partial \varphi}{\partial y}\right)^2}$$

$$\frac{uv}{a^2} \cong \frac{\left(U + \frac{\partial \varphi}{\partial x}\right)\frac{\partial \varphi}{\partial y}}{a'^2\left[1 - (\gamma-1)M'\frac{1}{a'}\frac{\partial \varphi}{\partial x} - \frac{\gamma-1}{2}\frac{1}{a'^2}\left(\frac{\partial \varphi}{\partial y}\right)^2\right]} \cong \frac{M'\frac{1}{a'}\frac{\partial \varphi}{\partial y}}{1 - (\gamma-1)M'\frac{1}{a'}\frac{\partial \varphi}{\partial x} - \frac{\gamma-1}{2}\frac{1}{a'^2}\left(\frac{\partial \varphi}{\partial y}\right)^2}$$

Therefore the differential equation becomes

$$\left[1-(\gamma-1)M^2\frac{1}{a^2}\frac{\partial\varphi}{\partial x}-\frac{\gamma-1}{2}\frac{1}{a^2}\left(\frac{\partial\varphi}{\partial y}\right)^2-M^2-2M^2\frac{1}{a^2}\frac{\partial\varphi}{\partial x}\right]\frac{\partial^2\varphi}{\partial x^2}-2M^2\frac{1}{a^2}\frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial x\partial y}$$

$$+\left[1-(\gamma-1)M^2\frac{1}{a^2}\frac{\partial\varphi}{\partial x}-\frac{\gamma-1}{2}\frac{1}{a^2}\left(\frac{\partial\varphi}{\partial y}\right)^2-\frac{1}{a^2}\left(\frac{\partial\varphi}{\partial y}\right)^2\right]\frac{\partial^2\varphi}{\partial y^2}=0$$

$$\left[1-(\gamma+1)M^2\frac{1}{a^2}\frac{\partial\varphi}{\partial x}-\frac{\gamma-1}{2}\frac{1}{a^2}\left(\frac{\partial\varphi}{\partial y}\right)^2-M^2\right]\frac{\partial^2\varphi}{\partial x^2}-2M^2\frac{1}{a^2}\frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial x\partial y}$$

$$+\left[1-(\gamma-1)M^2\frac{1}{a^2}\frac{\partial\varphi}{\partial x}-\frac{\gamma+1}{2}\frac{1}{a^2}\left(\frac{\partial\varphi}{\partial y}\right)^2\right]\frac{\partial^2\varphi}{\partial y^2}=0$$

Let $\varphi=a^2b\frac{1}{M^2}f(\xi,\eta)$

$x=b\xi$

$y=b\eta\left(\frac{t}{b}\right)^{n}$

Boundary conditions, at ∞, $\frac{\partial f}{\partial\xi}=\frac{\partial f}{\partial\eta}=0$

at $y=0$, $\left(\frac{\partial\varphi}{\partial y}\right)_{y=0}=a^2M^2\left(\frac{t}{b}\right)h(\xi)$

$$\left\{1-(\gamma-1)\frac{\partial f}{\partial\xi}-\frac{\gamma+1}{2}\frac{1}{\left[M^2\left(\frac{t}{b}\right)^{n}\right]^2}\left(\frac{\partial f}{\partial\eta}\right)^2\right\}\frac{\partial^2 f}{\partial\eta^2}=2\frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\xi\partial\eta}+\left[M^2\left(\frac{t}{b}\right)^{n}\right]^2\frac{\partial^2 f}{\partial\xi^2}$$

At $\eta=0$, $a^2b\frac{1}{M^2}\left(\frac{\partial f}{\partial\eta}\right)_{\eta=0}\frac{1}{b}\frac{1}{\left(\frac{t}{b}\right)^{n}}=a^2\left(M^2\frac{t}{b}\right)h(\xi)$

Or $\left(\frac{\partial f}{\partial\eta}\right)_{\eta=0}=\left[M^2\left(\frac{t}{b}\right)\right]\left[M^2\left(\frac{t}{b}\right)^{n}\right]h(\xi)$

Then if $M^2\left(\frac{t}{c}\right) = K$, then

$$\left\{1-(\gamma-1)\frac{\partial f}{\partial \xi} - \frac{\gamma+1}{2}\frac{1}{K^2}\left(\frac{\partial f}{\partial \eta}\right)^2\right\}\frac{\partial^2 f}{\partial \eta^2} = 2\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \xi \partial \eta} + K^2\frac{\partial^2 f}{\partial \xi^2}$$

Boundary conditions, At ∞, $\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial \eta} = 0$

At $\eta = 0$, $\left(\frac{\partial f}{\partial \eta}\right)_{\eta=0} = K^2 g'(\xi)$

$$p = p_0\left[1+\frac{\gamma-1}{2}\frac{M^2+2M\frac{\partial \varphi}{\partial x}+\left(\frac{\partial \varphi}{\partial y}\right)^2}{a^2\left[1-(\gamma-1)M\frac{1}{a}\frac{\partial \varphi}{\partial x}-\frac{\gamma-1}{2}\frac{1}{a^2}\left(\frac{\partial \varphi}{\partial y}\right)^2\right]}\right]^{-\frac{\gamma}{\gamma-1}}$$

$$p^0 = p_0\left[1+\frac{\gamma-1}{2}M^2\right]^{-\frac{\gamma}{\gamma-1}}$$

$$p = p^0\left[\frac{1+\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}\dfrac{M^2+2M\frac{1}{a}\frac{\partial \varphi}{\partial x}+\frac{1}{a^2}\left(\frac{\partial \varphi}{\partial y}\right)^2}{1-(\gamma-1)M\frac{1}{a}\frac{\partial \varphi}{\partial x}-\frac{\gamma-1}{2}\frac{1}{a^2}\left(\frac{\partial \varphi}{\partial y}\right)^2}}\right]^{\frac{\gamma}{\gamma-1}}$$

$$p = p^0\left[1-(\gamma-1)\frac{\partial f}{\partial \xi}-\frac{\gamma-1}{2}\frac{1}{K}\left(\frac{\partial f}{\partial \eta}\right)^2\right]^{\frac{\gamma}{\gamma-1}}$$

11)

$$D_{wing} = D = \int_{-b}^{b} p_\infty h(\xi)\left(\frac{t}{c}\right) dx = b p_\infty \left(\frac{t}{c}\right) \int_{-1}^{1} \left[1-(\gamma-1)\frac{\partial f}{\partial \xi} - \frac{\gamma-1}{2}\frac{1}{K^2}\left(\frac{\partial f}{\partial \eta}\right)^2\right]^{\frac{\gamma}{\gamma-1}} h(\xi)\, d\xi$$

$$C_D = \frac{D}{\frac{\rho}{2}U^2(2b)} = \frac{D}{\rho U^2 b} = \frac{\left(\frac{t}{c}\right)M^{*}}{M^{*3}}\frac{1}{\gamma}\int_{-1}^{1}\left[1-(\gamma-1)\frac{\partial f}{\partial \xi} - \frac{\gamma-1}{2}\frac{1}{K^2}\left(\frac{\partial f}{\partial \eta}\right)^2\right]^{\frac{\gamma}{\gamma-1}} h(\xi)\, d\xi$$

$$C_D = \frac{1}{M^{*3}}\mathcal{D}\left(M^{*}\frac{t}{c}\right)$$

$$C_L = \frac{1}{M^{*2}}\mathcal{L}\left(M^{*}\frac{\alpha}{b}\right)$$

Checks with the linear Ackeret theory!

Axially symmetric flows

$$\left[1-(\gamma+1)M^{*2}\frac{1}{a^*}\frac{\partial \varphi}{\partial x} - \frac{\gamma-1}{2}\frac{1}{a^{*2}}\left(\frac{\partial \varphi}{\partial y}\right)^2 - M^{*2}\right]\frac{\partial^2 \varphi}{\partial x^2} - 2M^{*2}\frac{1}{a^*}\frac{\partial \varphi}{\partial y}\frac{\partial^2 \varphi}{\partial x \partial y} + \left[1-(\gamma-1)M^{*2}\frac{1}{a^*}\frac{\partial \varphi}{\partial x} - \frac{\gamma+1}{2}\frac{1}{a^{*2}}\left(\frac{\partial \varphi}{\partial y}\right)^2\right]\frac{\partial^2 \varphi}{\partial y^2}$$

$$+\left[1-(\gamma-1)M^{*2}\frac{1}{a^*}\frac{\partial \varphi}{\partial x} - \frac{\gamma-1}{2}\frac{1}{a^{*2}}\left(\frac{\partial \varphi}{\partial y}\right)^2\right]\frac{1}{y}\frac{\partial \varphi}{\partial y} = 0.$$

Boundary conditions, at ∞, $\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial y} = 0$

At $y=1$, $\left(y\frac{\partial \varphi}{\partial y}\right)_{y=1} = a^* M^{*} f(\xi)\left(\frac{t}{c}\right)^2 b$

$$\left[1-(\gamma-1)\frac{\partial f}{\partial \xi} - \frac{\gamma-1}{2}\frac{1}{\left[M^{*}\left(\frac{t}{c}\right)^n\right]^2}\left(\frac{\partial f}{\partial \eta}\right)^2\right]\frac{\partial^2 f}{\partial \eta^2} + \left[1-(\gamma-1)\frac{\partial f}{\partial \xi} - \frac{\gamma-1}{2}\frac{1}{\left[M^{*}\left(\frac{t}{c}\right)^n\right]^2}\left(\frac{\partial f}{\partial \eta}\right)^2\right]\frac{1}{\eta}\frac{\partial f}{\partial \eta}$$

$$-2\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \xi \partial \eta} + \left[M^{*}\left(\frac{t}{c}\right)^n\right]^2\frac{\partial^2 f}{\partial \xi^2}$$

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$$\eta=0, \quad a^0 b \frac{1}{M^0}\left(\eta\frac{\partial f}{\partial \eta}\right)_{\eta=0} = a^0 M^0 \left(\frac{t}{b}\right)^2 b\, h(\xi)$$

$$\left(\eta\frac{\partial f}{\partial \eta}\right)_{\eta=0} = \left[M^0\left(\frac{t}{b}\right)\right]^2 h(\xi)$$

So

$$M^0\left(\frac{t}{b}\right) = K.$$

$$\left\{1-(\gamma-1)\frac{\partial f}{\partial \xi} - \frac{\gamma+1}{2}\frac{1}{K^2}\left(\frac{\partial f}{\partial \eta}\right)^2\right\}\frac{\partial^2 f}{\partial \eta^2} + \left\{1-(\gamma-1)\frac{\partial f}{\partial \xi} - \frac{\gamma-1}{2}\frac{1}{K^2}\left(\frac{\partial f}{\partial \eta}\right)^2\right\}\frac{1}{\eta}\frac{\partial f}{\partial \eta}$$

$$= 2\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \xi \partial \eta} + K^2\frac{\partial^2 f}{\partial \xi^2}$$

$$\text{At } \infty, \quad \frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial \eta} = 0.$$

$$\left(\eta\frac{\partial f}{\partial \eta}\right)_{\eta=0} = K^2 h(\xi)$$

$$C_D = \frac{1}{M^{0\,2}}\,\mathscr{D}\left(M^0\frac{t}{b}\right)$$

1.1.4.2

Similarity Laws for Non-steady Two-Dimensional Transonic and Hypersonic Flow

非定常二维跨声速和高超声速流动的相似律

这是作者已完成但未发表的"Similarity Laws for Non-steady Two-Dimensional Transonic and Hypersonic Flow"（非定常二维跨声速和高超声速流动的相似律）一文的初稿，共有11页。

作者在 Theodore von Kármán （冯·卡门）发表了跨声速流动相似律和他自己发表了高超声速流动相似律以后，进一步探讨了非定常流动的相似律问题。经过作者精巧的量纲分析，得到了下面两点很有意思的结论：

(1) 对於震荡机翼的跨声速流动来说，只要加上有关震荡的频率和幅值的两个相似性条件，即频率正比於 $(\delta/b)^{2/3}$ 以及幅值正比于 $(\delta/b)^{1/3}$（其中 δ 是机翼厚度，b 是翼展），那么只要保持相似参数 $K(=(1-M)(\delta/b)^{2/3})$ 的数值和定常跨声速流动所取的数值相同，震荡机翼绕流问题的相似律成立。

(2) 对於非定常高超声速流动来说，因为来流速度 U 很大，由翼展 b、圆频率 ω 和 U 组成的无量纲频率 $\Omega(=b\omega/U)$ 很小，因而问题可以当做准定常流动对待而大为简化，相似参数只有一个，也就是定常高超声速流动问题中的相似参数 $K(=M\delta/b)$。这一点是和跨声速情况不同的。

Non-Steady
Similarity Laws for Two-Dimensional Transonic
and Hypersonic Flows

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1. Introduction

The three important quantities which control the character of the flow of compressible fluid over a body are the free stream velocity U, the velocity of sound c^0 in the free stream and the lowest (spatial) propagation velocity of a signal in the free stream, $|c^0-U|$. That the last mentioned quantity should be an important variable is seen from the fact that it determines essentially the time in which a disturbance will travel across the surface of body. To make the linearization of the basic differential equation possible, we have to impose the condition that the disturbance created by the presence of the body must be small in comparison with all three of the basic quantities, U, c^0, $|c^0-U|$. In ordinary subsonic or supersonic flows all these three quantities are of the same order of magnitude and the body disturbance can be considered to be small to all these variables. A linearized theory for thin bodies is thus possible. But for transonic flows, the quantity $|c^0-U|$ is much smaller than either U or c^0. Then the disturbances of the body cannot be assumed to be small with respect to $|c^0-U|$. Therefore a linearized theory is not possible. For hypersonic flows, c^0 is much smaller than either U or $U-c^0$. Hence here also, no linearized theory is allowed.

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For transonic and for hypersonic flows then, we have to deal with the much more complicated non-linear equations. Th. von Kármán (Ref. 1) and the present author (Ref. 2) have shown, however, that certain similarity rules exist for these flows and the basic differential equations can be considerably simplified, although

still non-linear. It is the purpose of the present paper to extend the method to non-steady flows.

2. Transonic Flows

Let the free stream with velocity U be in the x-direction. The chord of the thin body occupies a part of the x-axis in the interval $-b \leq x \leq b$. c is the local velocity of sound and c^* is the critical velocity of sound when the fluid velocity is equal to c^*. The velocity potential $\varphi(x, y, t)$ for unsteady irrotational flow is defined through the velocity components u and v as follows:

$$u = c^* + \frac{\partial \varphi}{\partial x}, \qquad v = \frac{\partial \varphi}{\partial y} \tag{1}$$

The differential equation to be satisfied by φ is then

$$c^2\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right) = \frac{\partial^2 \varphi}{\partial t^2} + 2u\frac{\partial^2 \varphi}{\partial x \partial t} + 2v\frac{\partial^2 \varphi}{\partial y \partial t} + u^2\frac{\partial^2 \varphi}{\partial x^2} + 2uv\frac{\partial^2 \varphi}{\partial x \partial y} + v^2\frac{\partial^2 \varphi}{\partial y^2} \tag{2}$$

where c^2 is given by

$$c^2 = c^{*2} - \frac{\gamma - 1}{2}\left[2c^*\frac{\partial \varphi}{\partial x} + \left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + 2\frac{\partial \varphi}{\partial t}\right] \tag{3}$$

The boundary conditions at infinity have to be such as to give a flow of uniform velocity U along the x-axis. Therefore

$$\left.\begin{aligned} c^* + \frac{\partial \varphi}{\partial x} &= U \\ \frac{\partial \varphi}{\partial y} &= 0 \end{aligned}\right\} \text{ at } \infty \tag{4}$$

The boundary condition at the surface of the body has to be such as to give a fluid velocity tangential to the contour. Let the slope of the upper surface of the body be $\delta f_1(x, t)$ and the slope of the lower surface of the body be $\delta f_2(x, t)$; $-b \leq x \leq b$, and δ is the "thickness ratio" of the body, i.e., the maximum thickness of the body divided by the chord length $2b$. We are thus considering a family of similar bodies varying out lateral

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motions with displacement ~~proportional~~ a function of t. Since the body is considered to be thin and the fluid velocity deviates only slightly from c^*, we have the approximate boundary conditions at the surface of the body

$$\left.\begin{aligned}\left(\frac{\partial \varphi}{\partial y}\right)_{y=0+} &= c^* \delta\, g_1(x,t)\\ \left(\frac{\partial \varphi}{\partial y}\right)_{y=0-} &= c^* \delta\, g_2(x,t)\end{aligned}\right\}\; -b \le x \le b \qquad (5)$$

To obtain the similarity law, we introduce the following transformation of variables

$$x = b\xi, \quad y = b\,\frac{\eta}{\xi^n}, \quad t = \frac{b}{c^*}\,\frac{\tau}{(1-M^*)^{\kappa}}$$

$$\varphi = b c^* (1-M^*)^m f(\xi, \eta, \tau) \qquad (6)$$

(taken to be very close to 1)

where M^* is the free stream Mach number and m, n, κ are constants to be determined. Equation (2) now becomes

$$\frac{\partial^2 f}{\partial \eta^2} - (\gamma+1)\frac{(1-M^*)^{m}}{\xi^{2n}}\frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \xi^2} - 2\frac{(1-M^*)^{\kappa}}{\xi^{2n}}\frac{\partial^2 f}{\partial \xi \partial \tau}$$

$$= \frac{\gamma+1}{2}\frac{(1-M^*)^{2m}}{\xi^{2n}}\left(\frac{\partial f}{\partial \xi}\right)^2\frac{\partial^2 f}{\partial \xi^2} + (\gamma-1)(1-M^*)^{m}\frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \eta^2} + (\gamma-1)\frac{(1-M^*)^{m+\kappa}}{\xi^{2n}}\frac{\partial f}{\partial \tau}\frac{\partial^2 f}{\partial \xi^2}$$

$$+ \frac{(1-M^*)^{2\kappa}}{\xi^{2n}}\frac{\partial^2 f}{\partial \tau^2} + 2\frac{(1-M^*)^{m+\kappa}}{\xi^{2n}}\frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \xi \partial \tau} + 2(1-M^*)^{m}\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \xi \partial \eta}$$

$$+ \frac{\gamma+1}{2}(1-M^*)^{2m}\left(\frac{\partial f}{\partial \eta}\right)^2\frac{\partial^2 f}{\partial \xi^2} + \frac{\gamma-1}{2}(1-M^*)^{2m}\left(\frac{\partial f}{\partial \xi}\right)^2\frac{\partial^2 f}{\partial \eta^2} + (\gamma-1)(1-M^*)^{m+\kappa}\frac{\partial f}{\partial \tau}\frac{\partial^2 f}{\partial \eta^2}$$

$$+ (1-M^*)^{m+\kappa}\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \eta \partial \tau} + 2(1-M^*)^{2m}\frac{\partial f}{\partial \xi}\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \xi \partial \eta} + \frac{\gamma+1}{2}(1-M^*)^{2m}\xi^{2n}\left(\frac{\partial f}{\partial \eta}\right)^2\frac{\partial^2 f}{\partial \eta^2} \qquad (7)$$

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The boundary conditions specified by equations (4) and (5) now become

$$\left.\begin{aligned} (1-M^{\circ})^{m}\frac{\partial f}{\partial \xi} &= -\frac{2}{\gamma+1}(1-M^{\circ}) \\ \frac{\partial f}{\partial \eta} &= 0 \end{aligned}\right\} \text{ at } \infty \qquad (8)$$

and

$$\left.\begin{aligned} \frac{(1-M^{\circ})^{m}}{\delta^{1-n}}\left(\frac{\partial f}{\partial \eta}\right)_{\eta=0+} &= g(\xi,\tau) \\ \frac{(1-M^{\circ})^{m}}{\delta^{1-n}}\left(\frac{\partial f}{\partial \eta}\right)_{\eta=0-} &= g_{0}(\xi,\tau) \end{aligned}\right\} -1\le\xi\le 1 \qquad (9)$$

In order that the Mach number M° and thickness ratio δ do not explicitly enter into the system of equations (7), (8) and (9), we have from the first equation of (8),

$$m = 1 \qquad (10)$$

Then the appropriate value for α can be ascertained from equation (7) by the condition that the time variable τ must enter into the equation at least in one term which is of comparable magnitude to other terms retained in the equation. Then

$$\alpha = 1 \qquad (11)$$

With these values of m and α, terms on the right of equation (7) are all small compared with the terms on the left. Thus this equation is reduced to

$$\frac{\partial^2 f}{\partial \eta^2} = (\gamma+1)\frac{1-M^{\circ}}{\delta^{2n}}\frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \xi^2} + 2\frac{(1-M^{\circ})}{\delta^{2n}}\frac{\partial^2 f}{\partial \xi\partial \tau} \qquad (12)$$

Now by comparing equations (12) and (9), it is seen that

$$1-n = 2n$$

or

$$n = 1/3. \qquad (13)$$

Let
$$\frac{(1-M^2)}{\delta^{2/3}} = K_1 \tag{14}$$
then the differential equation (12) can be written as
$$\frac{\partial^2 f}{\partial \eta^2} = (\gamma+1) K_1 \frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \xi^2} + 2K_1 \frac{\partial^2 f}{\partial \xi \partial \tau} \tag{15}$$
The boundary conditions become
$$\left.\begin{aligned} \frac{\partial f}{\partial \xi} &= -\left(\frac{2}{\gamma+1}\right) \\ \frac{\partial f}{\partial \eta} & \end{aligned}\right\} \text{ at } \infty \tag{16}$$
and
$$\left.\begin{aligned} K_1\left(\frac{\partial f}{\partial \eta}\right)_{\eta=+0} &= g_1(\xi,\tau) \\ K_1\left(\frac{\partial f}{\partial \eta}\right)_{\eta=-0} &= g_2(\xi,\tau) \end{aligned}\right\} -1 \le \xi \le 1 \tag{17}$$
Equations (15), (16) and (17) form the system of reduced equations for transonic flows with a single similarity parameter K_1.

The ratio of specific heats γ can be also absorbed into the similarity parameter by making a slightly different transformation. In fact if
$$\varphi = c^* b \frac{1-M^2}{\Gamma} F(\xi,\zeta)$$
$$x = b\xi, \quad y = b\frac{\zeta}{(\delta\Gamma)^{1/3}}, \quad t = \frac{b}{c^*}\frac{\tau}{(1-M^2)} \tag{18}$$
with
$$\Gamma = \tfrac{1}{2}(\gamma+1) \tag{19}$$
the differential equation for $F(\xi,\zeta)$ is
$$\frac{\partial^2 F}{\partial \zeta^2} = 2K \frac{\partial F}{\partial \xi}\frac{\partial^2 F}{\partial \xi^2} + 2K \frac{\partial^2 F}{\partial \xi \partial \tau} \tag{20}$$
where
$$K = \frac{1-M^2}{(\delta\Gamma)^{2/3}} \tag{21}$$

The corresponding boundary conditions are then

$$\left.\begin{aligned}\frac{\partial F}{\partial \xi} &= -1\\ \frac{\partial F}{\partial \zeta} &= 0\end{aligned}\right\} \quad \text{at} \quad \infty \tag{22}$$

and

$$\left.\begin{aligned}K\left(\frac{\partial F}{\partial \zeta}\right)_{\zeta=0+} &= g_1(\xi, \tau)\\ K\left(\frac{\partial F}{\partial \zeta}\right)_{\zeta=0-} &= g_2(\xi, \tau)\end{aligned}\right\} \quad -1 \leq \xi \leq 1 \tag{23}$$

For steady flows, the system of equations (20), (21), (22) and (23) reduces to that of von Kármán (Ref. 1).

The static pressure p at any point can be readily calculated by using the Bernoulli equation

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}\left\{\left(c^* + \frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2\right\} + \frac{\gamma}{\gamma-1}\frac{p}{\rho} = \frac{1}{2}U^2 + \frac{p^0}{\rho^0} \tag{24}$$

and the isentropic relation. By retaining only terms of first order of magnitude, we find the pressure coefficient C_p as

$$C_p = \frac{p - p^0}{\frac{1}{2}\rho^0 U^2} = -\frac{2(1-M^2)}{\Gamma}\left(1 + \frac{\partial F}{\partial \xi}\right) \tag{25}$$

Or

$$C_p = -\frac{\delta^{2/3}}{\Gamma^{1/3}}\mathcal{P}(K, \xi, \tau) \tag{26}$$

Integration of the pressure coefficient over the surface of the body yields the drag coefficient C_d and the lift coefficient C_l. These are

$$C_d = \frac{\delta^{5/3}}{\Gamma^{1/3}}\mathcal{D}(K, \tau) \tag{27}$$

and

$$C_l = \frac{\delta^{2/3}}{\Gamma^{1/3}}\mathcal{L}(K, \tau) \tag{28}$$

Now let us consider the specific problem of an symmetrical airfoil at zero angle of attack in lateral oscillation. Let the slope of the upper and lower surface of the airfoil be $\delta g(\xi)$ and $-\delta g(\xi)$ respectively. And the amplitude of oscillation be a. Then the lateral velocity due to oscillation can be written as $a\omega \sin\omega t$. Hence

$$c^* \delta g_1 = c^* \delta g(\xi) + a\omega \sin\omega t \qquad (29)$$

By introducing the reduced frequency Ω defined as

$$\Omega = \frac{b}{c^*}\frac{\omega}{(1-M^*)} = \frac{b\omega}{c^*(\delta\Gamma)^{2/3}}\frac{1}{K} \qquad (30)$$

equation (29) can be re-written as

$$g_1(\xi,\tau) = g(\xi) + \left(\frac{a}{b}\right)\frac{\Gamma^{2/3}}{\delta^{1/3}} K\Omega \sin\Omega\tau \qquad (31)$$

Similarly

$$g_2(\xi,\tau) = -g(\xi) + \left(\frac{a}{b}\right)\frac{\Gamma^{2/3}}{\delta^{1/3}} K\Omega \sin\Omega\tau \qquad (32)$$

Therefore the flows of the same [illegible] over the family of airfoil will be similar under the condition that the similarity parameter K is a constant, if

1) $(1-M^*)$ is proportional to $\delta^{2/3}$
2) the circular frequency ω is proportional to $(1-M^*)$ or $\delta^{2/3}$

and 3) the amplitude ratio $\left(\frac{a}{b}\right)$ is proportional to $(1-M^*)^{1/2}$ or $\delta^{1/3}$, i.e., the velocity amplitude $a\omega$ is proportional to δ.

The fact that the reduced frequency Ω has the form given by equation (30) agrees with the result of linearized theory. For instance, for supersonic flow with $M>1$, the reduced frequency Ω' (Ref. 3) is

$$\Omega' = \frac{\omega b}{U}\frac{2M^2}{M^2-1} \qquad (33)$$

For transonic flow $U \sim c^*$, $M^* \sim 1$, $M^2-1 \sim 2(M^*-1)$. Then Ω' reduces to Ω. In engineering applications, this form of

2

reduced frequency has the meaning that in transonic flow the effective frequency is the real frequency ω multiplied by the factor $1/(1-M^2)$. We know that at high effective frequencies, the aerodynamic forces acting on the airfoil cannot be calculated by the quasi-steady method and they have large out of phase components. Therefore when one is investigating the stability problems of aircrafts in transonic flight, this peculiarity of transonic non-steady flow must be kept in mind.

3. Hypersonic Flow

Let the free stream velocity be U in the direction of the x-axis. The body occupies a section of the x-axis in the interval $-b \leq x \leq b$. The disturbance velocity potential $\varphi(x, y, t)$ is defined through the velocity components u and v by

$$u = U + \frac{\partial \varphi}{\partial x}, \qquad v = \frac{\partial \varphi}{\partial y} \tag{34}$$

The differential equation for φ is then

$$c^2\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right) = \frac{\partial^2 \varphi}{\partial t^2} + 2u\frac{\partial^2 \varphi}{\partial x \partial t} + 2v\frac{\partial^2 \varphi}{\partial y \partial t} + u^2\frac{\partial^2 \varphi}{\partial x^2} + 2uv\frac{\partial^2 \varphi}{\partial x \partial y} + v^2\frac{\partial^2 \varphi}{\partial y^2} \tag{35}$$

where c is the local velocity of sound given by

$$c^2 = U^2\left[\frac{1}{M^2} - \frac{\gamma - 1}{2}\left\{2\frac{1}{U}\frac{\partial \varphi}{\partial x} + \frac{1}{U^2}\left(\frac{\partial \varphi}{\partial x}\right)^2 + \frac{1}{U^2}\left(\frac{\partial \varphi}{\partial y}\right)^2 + 2\frac{1}{U^2}\frac{\partial \varphi}{\partial t}\right\}\right] \tag{36}$$

M is the Mach number of the free stream and is assumed to be large. The boundary conditions at infinity are

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial y} = 0 \qquad \text{at } \infty \tag{37}$$

At the surface of the body,

$$\left(\frac{\partial \varphi}{\partial y}\right)_{\text{surface}} = U\delta\, h_{\text{surface}}(x, t), \tag{38}$$

where δ is the thickness ratio of the body, here assumed to be small

By using a procedure similar to that explained in the previous section for the transonic flow, the appropriate parameters of transformation are found to be

$$x = b\xi, \quad y = b\delta\eta, \quad t = \frac{b}{U}\tau$$
$$\varphi = bU\frac{1}{M^2} f(\xi, \eta, \tau) \qquad (39)$$

Then the differential equation (38) is reduced to

$$\left[1 - (\gamma-1)\left(\frac{\partial f}{\partial \xi} + \frac{\partial f}{\partial \tau}\right) - \frac{\gamma+1}{2}\frac{1}{K^2}\left(\frac{\partial f}{\partial \eta}\right)^2\right]\frac{\partial^2 f}{\partial \eta^2} = K^2\left[\frac{\partial^2 f}{\partial \tau^2} + 2\frac{\partial^2 f}{\partial \xi \partial \tau} + \frac{\partial^2 f}{\partial \xi^2}\right]$$
$$+ 2\frac{\partial f}{\partial \eta}\left(\frac{\partial^2 f}{\partial \eta \partial \tau} + \frac{\partial^2 f}{\partial \xi \partial \eta}\right) \qquad (40)$$

where the similarity parameter K is given by

$$K = M\delta \qquad (41)$$

The boundary conditions are now

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial \eta} = 0 \quad \text{at } \infty \qquad (42)$$

and

$$\left(\frac{\partial f}{\partial \eta}\right)_{\text{body}} = K^2 h_\xi(\xi, \tau) \qquad (43)$$

This set of equations (40), (41), (42) and (43) together with the transformation equation (39) constitute the basis of hypersonic flow over a thin body. They reduce to the equations previously obtained (16/2) when the motion is steady.

The pressure coefficient C_p is now

$$C_p = \frac{2}{\gamma}\frac{1}{M^2}\left[\left\{1 - (\gamma-1)\left(\frac{\partial f}{\partial \xi} + \frac{\partial f}{\partial \tau}\right) - \frac{\gamma-1}{2}\frac{1}{K^2}\left(\frac{\partial f}{\partial \eta}\right)^2\right\}^{\frac{\gamma}{\gamma-1}} - 1\right] \qquad (44)$$

Or

$$C_p = \delta^2 \mathcal{P}(K, \gamma; \xi, \tau) \qquad (45)$$

By integrating the pressure over the surface of the body, we obtain the drag coefficient C_d and the lift coefficient C_l as

$$C_d = \delta^3 \mathcal{D}(K, \gamma; \tau) \qquad (46)$$

and

$$C_l = \delta^2 \mathcal{L}(K, \gamma; \tau) \qquad (47)$$

The transformation of the time variable as given by equation (39) shows that for an oscillatory motion of the body, the reduced frequency Ω is

$$\Omega = \frac{b\omega}{U} \qquad (48)$$

This again agrees with the results of the linearized theory given by equation (33), as $M^2/M^2-1 \to 1$ if M is large. In engineering applications, this form of reduced frequency means that the effective frequency will be much smaller than the real frequency as the free stream velocity U is very large for hypersonic flows. This is contrary to the transonic flows. For this case then, it is possible that a quasi-steady calculation is generally sufficient and thus greatly simplify the problem.

References

1) Th. von Kármán: "The Similarity Law of Transonic Flow" J. of Math. & Phy. Vol. 26, pp. 182-190 (1947)

2) H. S. Tsien, "Similarity Laws of Hypersonic Flows" J. of Math. & Physics, Vol. 25, pp. 247-251 (1946)

3) See for instance, I. E. Garrick and S. I. Rubinow, "Flutter and Oscillating Air-Force Calculations for an Airfoil in a Two-dimensional Supersonic Flow" NACA Tech. Note No. 1158 (1946)

1.1.5
风洞设计

1.1.5.1
弹道试验用超声速风洞的设计

1940 年 6 月，加州理工学院的古根海姆航空实验室为了要建造一个弹道试验用的超声速风洞，实验室负责人 Theodore von Kármán(冯·卡门)委托钱学森为这一设计进行全面的方案论证和分析计算。分析计算的手稿长达 143 页，与手稿一起附有一份风洞设计的说明书(打印稿，9 页)，名为“Memorandum on Supersonic Wind Tunnel for Ballistic Purposes”(关于弹道试验用超声速风洞的备忘录)。

为设计闭合循环连续工作的超声速风洞，钱学森考虑了两类可能的驱动气流的方案，即“Direct operation”(直接驱动)和“Induction operation”(引射驱动)。采用直接驱动，风洞中的气流直接由压气机不断提供，优点是流经试验段气流的动能与输入功有较高的比值。采用引射驱动，压气机所需供给的气流量比流经试验段的气流量要小，而且主气流稳定性好。作者吸取了两者的优点，决定采用直接驱动和引射驱动的组合方案，在低速情况(马赫数 $M<1.2$)采用引射驱动，而在高速情况($1.2<M<3.2$)采用直接驱动。这样的组合既可满足炸弹试验的低速要求，也能适应其他弹道试验的高速需要。

作者的手稿中，围绕上述两种驱动方式进行了不同工况下的参数(试验段断面积、气流马赫数、压缩比、体积流量等)计算，风洞设计部分涉及到压气机、冷却器、试验段设备(天平、纹影仪、真空泵等)、风洞结构(管路、喷管、试验段)等的配置以及其他工程问题(风洞模型试验、工程设计、标定等)的解决方案。

这里选印了分析计算手稿中的 6 页，即手稿的第 1—4 页和第 9 页以及

风洞设计说明书的首页。作者在这一部分手稿中针对直接驱动方案，进行了主要参数的计算，其结果反映在手稿第 9 页上的压缩比 λ 与输入空气的体积流率 V_0 的变化关系中。

从这份手稿中，我们可以看到，钱学森不仅是一位卓越的工程科学家，而且曾经主持过超声速风洞这样的大工程的设计；也正是由于他具有工程设计方面的丰富经验，才能切实把握工程技术的实际需要，为发展与航空有关的技术科学做出杰出的贡献。他在国外所奠定的工程科学理论和工程设计经验，成为他回国以后在技术上领导我国火箭导弹和航天事业的基础。

PRELIMINARY DESIGN

1

(I) If the air in entrance cone is under following conditions:

pressure ~ p_1

density ~ ρ_1

temperature ~ 580° F. Abs.

and the air in test section is under the following conditions:

pressure ~ p_t

density ~ ρ_t

temperature ~ T_t,

the following relation exists (neglecting velocity in the entrance cone)

$$\frac{p_1}{p_t} = \left\{1 + \frac{\gamma-1}{2}\left(\frac{v_t}{a_t}\right)^2\right\}^{\frac{\gamma}{\gamma-1}}$$

$$\frac{T_1}{T_t} = \left(\frac{p_1}{p_t}\right)^{\frac{\gamma-1}{\gamma}} = 1 + \frac{\gamma-1}{2}\left(\frac{v_t}{a_t}\right)^2$$

$$\frac{\rho_1}{\rho_t} = \frac{p_1}{p_t}\cdot\frac{T_t}{T_1} = \left\{1 + \frac{\gamma-1}{2}\left(\frac{v_t}{a_t}\right)^2\right\}^{\frac{1}{\gamma-1}}$$

Therefore if we put $\frac{v_t}{a_t} = M_t$, and $\gamma = 1.405$,

$$\frac{p_1}{p_t} = (1 + 0.2025 M_t^2)^{3.470}, \qquad \frac{T_1}{T_t} = 1 + 0.2025 M_t^2$$

$$\frac{\rho_1}{\rho_t} = (1 + 0.2025 M_t^2)^{2.470}$$

2

$$a = 49.1\sqrt{T} \quad \text{ft/sec.}$$

$$\text{Horsepower} = \frac{\frac{\rho_t}{2} v_t^3 A_t}{550 \times \text{Energy Ratio}} = 3000 \text{ [illegible]}$$

Let λ = compression ratio,

$$\lambda = \left(\frac{T_1}{T_t}\right)^{\frac{\gamma}{\gamma-1}(1-\eta_D)} = \left(1 + 0.2025 M_t^2\right)^{3.470(1-\eta_D)}$$

M_t	$1+0.2025M_t^2$	T_t, Abs.	a_t	v_t	Energy Ratio*	$\rho_t A_t$	η_D †
1.20	1.2914	449	1039	1247	2.60	0.00443	0.750
1.[illegible]	1.3968	415	1000	1400	2.49	0.002995	0.748
1.60	1.518	382	959	1534	2.34	0.002140	0.745
1.80	1.656	350	917	1650	2.15	0.001578	0.740
2.00	1.810	320.4	875	1750	1.92	0.001180	0.730
2.20	1.980	292.8	838	1845	1.70	0.000883	0.715
2.40	2.165	268.0	804	1929	1.47	0.000675	0.693
2.60	2.368	244.8	769	1998	1.26	0.000521	0.661
2.80	2.587	224.2	735	2058	1.11	0.000420	0.643
3.00	2.821	205.5	703	2108	0.99	0.000348	0.625
3.20	3.072	188.7	674	2154	0.90	0.000297	0.613

$$\rho_t A_t = \frac{\text{Energy Ratio} \times 3{,}000 \times 1100}{v_t^3}$$

* L. Crocco: p. 71. † L. Crocco: p.

M_t	$3.470(1-\eta_0)$	λ	$\frac{1}{\gamma-1}-\frac{\gamma(1-\eta_0)}{\gamma-1}$	$\frac{\rho_0}{\rho_t}$	$\rho_t A_t v_t$		
1.20	0.867	1.248	1.603	1.507	5.53		
1.40	0.874	1.339	1.596	1.704	4.19		
1.60	0.885	1.446	1.585	1.938	3.262		
1.80	0.902	1.576	1.568	2.207	2.605		
2.00	0.937	1.743	1.533	2.483	2.065		
2.20	0.988	1.964	1.472	2.732	1.627		
2.40	1.064	2.225	1.406	2.962	1.301		
2.60	1.150	2.695	1.320	3.120	1.040		
2.80	1.237	3.240	1.233	3.224	0.813		
3.[illegible]	1.300	3.850	1.170	3.366	0.734		
3.20	1.341	4.510	1.129	3.550	0.640		

Let the density, pressure, temperature at inlet to the compressor be $\rho_0,\ p_0,\ T_0$.

$$\text{Mass flow} = \rho_t A_t v_t = \rho_0 A_0 v_0$$

$$A_0 v_0 = V_0 = \frac{\rho_t A_t v_t}{\rho_0} = \left(\frac{\rho_t A_t v_t}{\rho_t}\right)\frac{\rho_t}{\rho_0}$$

But
$$\frac{\rho_t}{\rho_0} = \frac{p_t}{p_0}\cdot\frac{T_0}{T_t} = \left(\frac{p_t}{p_0}\right)^{-\frac{\gamma-1}{\gamma\eta_0}+1} = \left(\frac{p_t}{p_0}\right)^{\frac{1-\gamma(1-\eta_0)}{\gamma\eta_0}}$$

But $\frac{p_t}{p_0} = \left(\frac{\rho_t}{\rho_0}\right)^{\eta_0}$. $\therefore$
$$\frac{\rho_0}{\rho_t} = \left(1+0.2015 M_t^2\right)^{\frac{1}{\gamma-1}-\frac{\gamma(1-\eta_0)}{\gamma-1}}$$

4

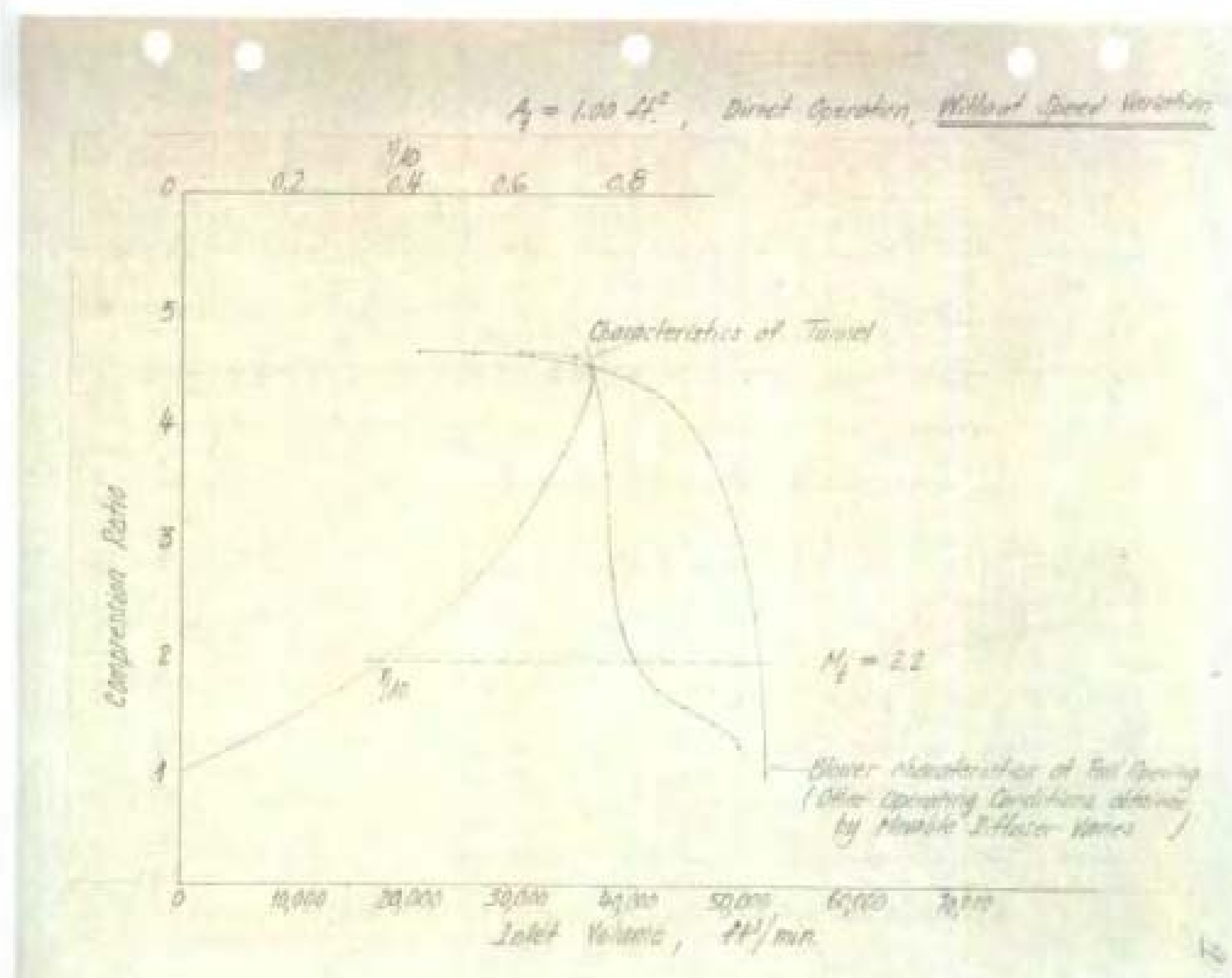
$A_t = 1$ м²

M_t	ρ_t	V_0 м³/мин	p_t, атм.	p_s/p_t	p_0/p_t	p_0, атм.
1.20	0.00443	49,700	1.611	2.430	1.947	3.140
1.40	0.002995	49,300	1.107	3.188	2.380	2.635
1.60	0.002140	47,500	0.662	4.26	2.945	1.947
1.80	0.001578	44,900	0.448	5.76	3.655	1.637
2.00	0.001180	42,300	0.3064	7.84	4.500	1.378
2.2	0.000883	40,500	0.2095	10.72	5.46	1.143
2.4	0.000675	39,100	0.1466	14.60	6.42	0.941
2.6	0.000521	38,400	0.1033	19.90	7.40	0.765
2.8	0.000420	38,000	0.0763	27.1	8.36	0.638
3.0	0.000348	37,600	0.0580	36.7	9.53	0.553
[illegible]	0.000297	36,400	0.0455	49.3	10.93	0.497

$$\frac{p_s}{p_t} = (1 + 0.2025\, M_t^2)^{3.440}$$

$$\frac{p_0}{p_t} = \frac{1}{\lambda}(1 + 0.2025\, M_t^2)^{3.440}$$

$$p_0 = \frac{p_t}{\lambda}(1 + 0.2025\, M_t^2)^{3.440}$$



Direct Operation,
0
0.2
0.4
0.6
0.8
Characteristics of Tunnel
Compression Ratio
1
2
3
4
5
M = 2.2
0
10,000
20,000
30,000
40,000
50,000
60,000
70,000
Inlet Volume, ft³/min

MEMORANDUM ON SUPERSONIC WIND TUNNEL
FOR BALLISTIC PURPOSES

1. Methods of Operation.

Supersonic wind tunnels with continuous operation in closed circuits can be classified in two systems:

a) In wind tunnels with "direct operation" the airstream velocity is maintained by passing through a compressor.

b) In wind tunnels of the "induction system" the compressor creates a jet of high speed which maintains the circulation of the main airstream by ejector action.

The energy ratio - i.e., the ratio between the rate at which kinetic energy passes through the test section and the power input necessary - is in general more favorable for the first mentioned system. However, the induction system has several advantages, especially in the velocity range near the speed of sound. First, the air volume handled by the compressor is small compared to the volume passing through the test section, and second, the stability of the main airstream is favorably influenced by the driving jet. For these reasons a combination of the two systems is suggested using the ejector drive for lower velocities (Mach's number < 1.2) and the direct operation for higher velocities up to the maximum Mach's number of about 1.2. This combination makes it possible to use a test chamber of larger cross section and correspondingly larger models in the lower velocity range. This range is especially important for problems of bomb design, whereas the problems of shell design and other problems of exterior ballistics require the higher velocity range.

2. Compressor Characteristics.

a) Direct operation.

Assuming a certain test section and the lowest practicable value of the

1.1.5.2

Proposal and Study for the Construction of a Pilot Hypersonic Wind Tunnel at the Massachusetts Institute of Technology

关于在麻省理工学院建造中间规模的高超声速风洞的建议和研究报告

1947年3—5月，钱学森和J. R. Markham、M. Witunski联名向麻省理工学院航空工程系的领导送交一份建议和研究报告，名为“Proposal and Study for the Construction of a Pilot Hypersonic Wind Tunnel at the Massachusetts Institute of Technology”（关于在麻省理工学院建造中间规模的高超声速风洞的建议和研究报告）。他们充分认识到发展火箭技术的重要性，它将大大推动空气动力学的发展。他们甚至考虑到了，在不久的将来将会实现宇宙飞行，火箭的最大飞行速度将大大超过4 km/s，相应的马赫数将远大于13。从火箭发射到脱离地球的重力场，人们对火箭所受的空气动力作用最不清楚的大概是在80 km左右的高度，在这样的高度上火箭的飞行马赫数大约在7—13之间，这属于高超声速的范围。为此，需要一个高超声速风洞以便开展必要的实验研究工作。然而，在建造这种大型风洞之前，必须解决风洞设计中的许多难题。鉴于当时在美国只有加州理工学院有一个小尺寸的高超声速风洞，其试验段中的气流难以得到均匀的速度。于是，作者建议在麻省理工学院建造一个中间规模的高超声速风洞，以便利用这个风洞开展实验研究，解决为设计大型高超声速风洞所遇到的难题。

在钱学森的手稿档案中保存有上述建议和研究报告的打印稿，共26页。和这一报告放在一起的还有一份钱学森所写的高超声速风洞扩压器的设计书，包括正文13页、附图4页和计算手稿12页。作者以其巧妙的构思，设计了一种新型的扩压器——楔型扩压器，当流过试验段的高超声速气流通过扩压器时，将会产生一系列的斜击波，使气流马赫数逐级降下来，从而

得到满足需要的压缩比，可以大大改善扩压器的压力恢复的性能，满足高超声速风洞的要求。

这里选印了上述建议和研究报告的最前面3页和1张风洞示意图，以及扩压器计算手稿的前4页和扩压器的示意图。

-1-

SUMMARY

It is proposed to construct a pilot hypersonic wind tunnel at the Supersonic Laboratory of the Massachusetts Institute of Technology of the following characteristics:

1. Intermittent operation of "push-pull" type with the tunnel located between a high pressure tank and a vacuum tank.
2. Test Section Mach number 10.
3. Test Section Cross-section of 500 square inches or 23 inches square.

A preliminary cost estimate shows that the total expenditure for design and construction of this pilot hypersonic wind tunnel to be $387,000.

The planned small research group to be specifically attached to this pilot hypersonic wind tunnel consists of:

1 Research Associate
2 Research Assistants
2 Assistants

CONTENTS

I. THE TREND TOWARDS HYPERSONIC VELOCITIES

While the need for research in the field of supersonic aerodynamics was not apparent a few years ago, there is no doubt today in the imortance of this field of research. However, fundamental research, in differentiation from developmental research, cannot stop at the present need, but must look beyond and anticipate future trends. What, then, is the trend of future advance in the field of aerodynamics?

It is certain that the range of rockets will increase. The most advanced rocket today is the V-2 rocket which has a range of approximately 200 miles and a maximum velocity of 3500ft. per sec. Since the range of a rocket of the V-2 type is approximately proportional to the square of the maximum flight velocity, a range of 3000 miles requires a maximum flight velocity of approximately 13,000 ft. per sec. The corresponding Mach number would be roughly 13. If one is to look even further, one must consider the probability and the possibility of satellite rockets and rockets capable of leaving the gravitational field of the earth These rockets must have a maximum velocity much greater than 13,000 ft. per sec. However, it will be also true that the maximum velocities of these rockets will be reached only at extreme altitudes (of the order of 100 miles) where the density of the atmosphere is so low as to make the aerodynamical forces negligible. Therefore, the most important range of speed to be investigated is the range corresponding to flight altitudes much lower than 100 miles or approximately 50 miles. At this altitude,

they

LONGITUDINAL VIEW

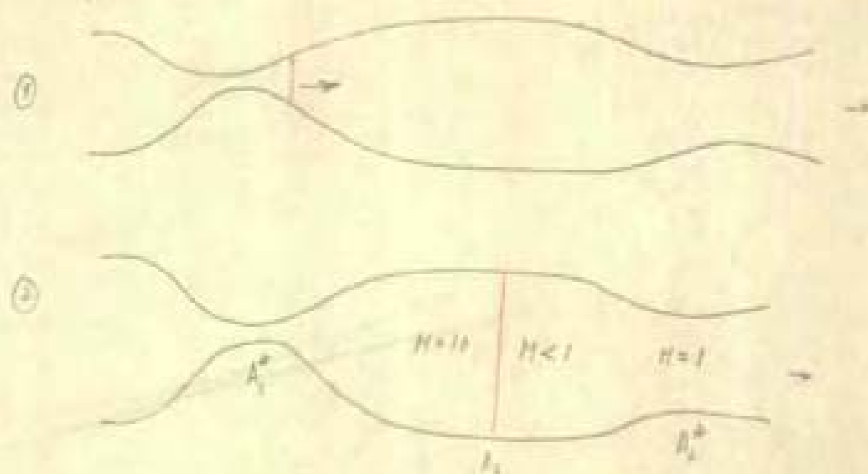
VACUUM PUMP

Investigation of the Diffuser

1

Hardcastle Type

Let us assume that the operation of the wind tunnel is started by a very large suction in the following way.



We first calculate the pressure ratio $\frac{p_0}{p_1} = (1+0.2\times 10^2)^{3.5}$

$$= 21^{3.5} = (2\times 10.5)^{3.5}$$

$$= 11.30\times 3760 = 42,500$$

Area ratio $\frac{A}{A_1^*} = \frac{1}{M}\left(\frac{1+0.2M^2}{1.2}\right)^3 = \frac{1}{10}\left(\frac{21}{1.2}\right)^3 = 535$

Mach number after normal shock at test section,

$$M_2^2 = \frac{1+\frac{\gamma-1}{2}M_1^2}{\gamma M_1^2 - \frac{\gamma-1}{2}} = \frac{21}{140-0.2} = 0.15$$

$$M_2 = 0.387$$

So $\frac{A_1}{A_2^*} = \frac{1}{0.387}\left(\frac{1.03}{1.2}\right)^3 = 1.591$; $\frac{A_2^*}{A_1^*} = \frac{535}{1.591} = 336$

2

Now if $M = 7$,

$$\frac{A}{A^*} = \frac{1}{7}\left(\frac{10.8}{1.2}\right)^2 = \frac{9}{7}\, 81 = 104$$

if $M = 8$

$$\frac{A}{A^*} = \frac{1}{8}\left(\frac{13.8}{1.2}\right)^2 = 192.5$$

if $M = 9$,

$$\frac{A}{A^*} = \frac{1}{9}\left(\frac{17.2}{1.2}\right)^2 = 319$$

So the Mach number at the second throat after the normal shock has been swallowed is 9.04

Now if we do not have the second throat:

$$\frac{p_s}{p_t} = \frac{2\gamma}{\gamma+1}10^2 - \frac{\gamma-1}{\gamma+1} = \frac{2.8}{2.4}100 - \frac{0.4}{2.4} = 116.6 - 0.17 = 116.4$$

Hypothetical compression ratio without second throat $= \frac{42{,}500}{116.4} = 365$

With second throat:



$$\frac{p_0}{p_2'} = (1 + 0.2 \times 9.04^2)^{3.5} = 17.35^{3.5} = 21{,}700$$

$$\frac{p_2}{p_2'} = \frac{2.8}{2.4}\, 9.04^2 - 0.17 = 95.2 - 0.17 = 95.0$$

2

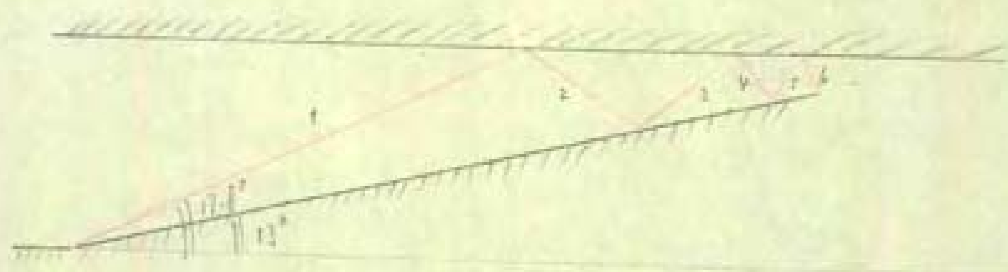
Hypothetical compression ratio with second throat of Kantrowitz type

$$= \frac{21,700}{95} = 228$$

Wedge Diffuser (Using chart for $k = 1.405$)

Take angle of deflection = 13°

No. of Shock	M_1	M_2	$M_1 \sin\gamma$	$\frac{2\gamma}{\gamma+1}(M_1 \sin\gamma)^2 - \frac{\gamma-1}{\gamma+1}$
1	10	5.75	3.05	10.69
2	5.75	4.12	2.06	4.70
3	4.12	3.16	1.72	3.30
4	3.16	2.47	1.54	2.60
5	2.47	1.95	1.41	2.15
6	1.95	1.50	1.35	1.96



Final normal shock $\frac{2\gamma}{\gamma+1}(1.50)^2 - \frac{\gamma-1}{\gamma+1} = 2.08$

Overall compression ratio of diffuser = $10.69 \times 4.70 \times 3.30 \times 2.60 \times 2.15 \times 1.96 \times 2.08$

$= 3860$

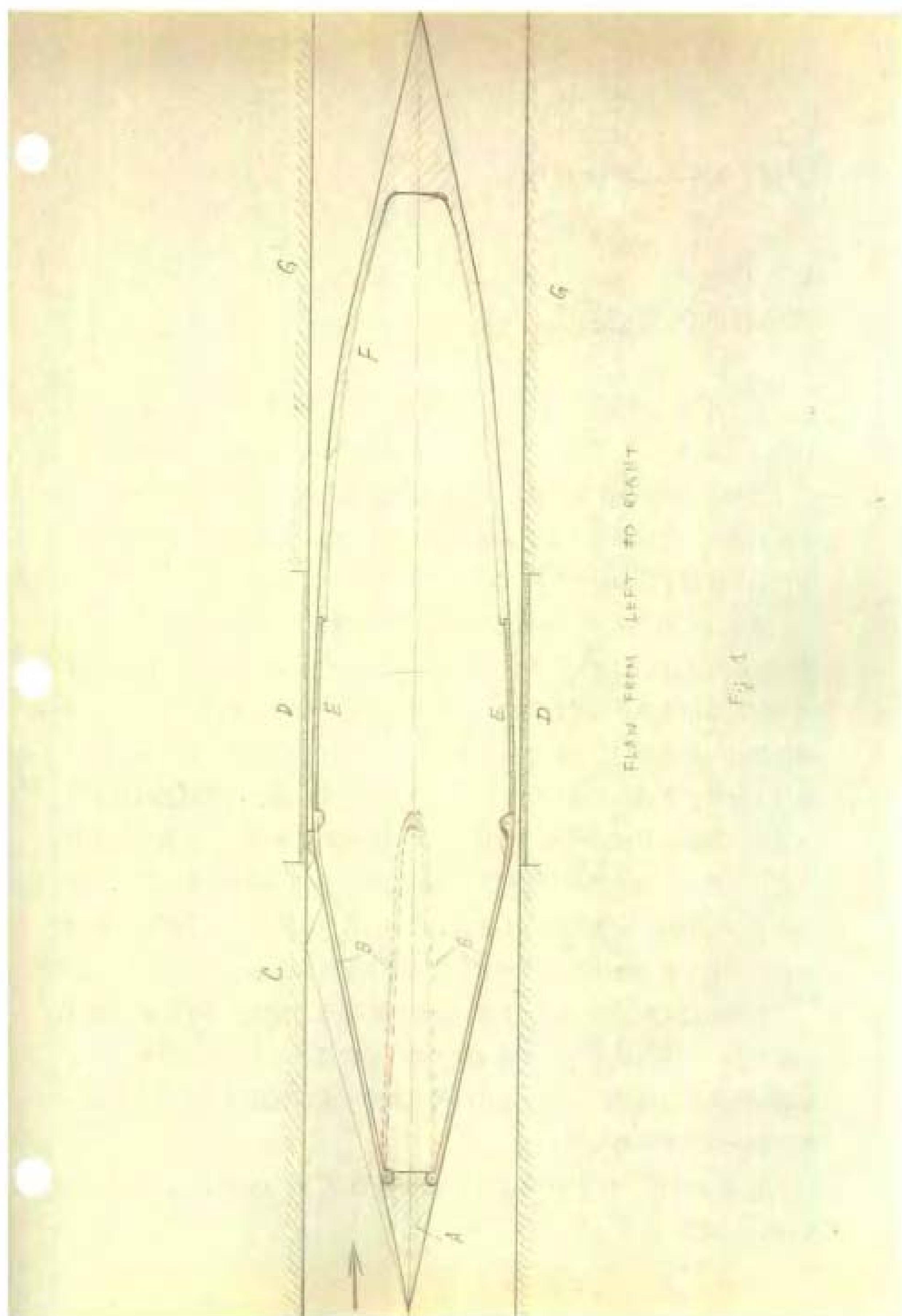
Hypothetical compression ratio of wedge diffuser $= \frac{42,500}{3860} = 11$

9

Final Results

Type of Diffuser	Nominal Compression Ratio req'd for M=10
Normal shock	365
Kantrowitz (fixed contour starting)	228
Wedge with 13° wedge angle	11

This shows that enormous improvement of hypersonic diffuser is possible by using a system of oblique shock waves. The case calculated consists of 6 oblique shocks and one normal shock. However this system will not be self-starting as the second throat is too small. The minimum second throat is given by the Kantrowitz condition. So a trick starting method is necessary.



G
G
F
D
E
E
D
FLOW FROM LEFT TO RIGHT
C
B
B
A

1.2
固体力学——壳体屈曲

1.2.1
壳体屈曲的文献总结

这是作者为研究壳体屈曲问题而对前人工作所做的总结和评述，共有18页，未发表。

作者在1939年6月取得博士学位后，任加州理工学院的助理研究员，开始对薄壳的失稳问题发生了兴趣。在深入研究这一问题之前，作者首先对前人的工作做了系统的总结和评述。

作者之所以选择这一课题进行研究，出自两方面的考虑。其一，当时第二次世界大战已经开始，飞机在战争中的重要作用益发明显，各国正在设计和制造全金属薄壳形式的飞机。薄壳结构的强度高而重量轻，当其接受的载荷超过一定数值时，壳体会发生皱瘪而失效，称之为屈曲。设计师需要知道发生这种屈曲的临界载荷的大小，可是用经典线性理论计算得到的数值却远高于试验值，只能依赖从相当分散的试验数据中整理得到的经验关系来进行设计。其二，为了解决上述理论和实验之间的矛盾，困难很大。理论上必须放弃小变形假设，需要考虑大挠度的影响，数学上遇到求解非线性方程的困难；实验上对条件的控制和现象的观测要求有高的技术。

作者通过文献总结，剖析了前人理论的优缺点，利用了当时可能得到的实验数据，认为应该从考虑有限挠度的弹性屈曲理论入手，采用能量法求取屈曲临界载荷。这里作者已经勾画出他后来陆续发表的数篇经典论文中提出的能量跃变准则的初步轮廓。

在文章结尾，作者说明文中的观点吸收了 W. L. Holland 和 E. E. Sechler 的建议。

1)

Report on the Present State of Theory of Thin Plates and Cylindrical Shells in Compression

H.S. Tsien

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5,

(I)

General Consideration

~~Generally speaking~~, ~~there~~ From the point of view of method of ^design^ analysis, there are two types of monocoque construction. One ~~type~~ which occurs in rather small aircraft has ~~rather~~ strong ~~stiffener~~ ~~and~~ rings – weak skin covering, and one which ~~occurs~~ appears more recently due to the increase in size of aircraft has ~~rather~~ weak rings. For the first type of structures, the buckling always occurs between the rings and the stiffened sheet can be considered as simply supported between the rings. The current American practice ~~is to~~ assumes the validity of Navier beam hypothesis and ~~use for the~~ determine the effective moment of inertia of the section by means of effective width of the sheet. H. Wagner (9) and H. Ebner and H. Köller (11) suggested the method of taking the system of stiffeners + rings as an statically indeterminate truss with sheet ^which is assumed only taking the^ replacing the diagonal bracing. ~~[struck]~~ Thus the problem is reduced to that of ^rigid^ airship structure. However, from the experience of Wagner tension field beam theory, this

method of analysis tends to give very conservative 6)
& heavy structure.

The second type of structure which has weak rings, ~~usually buckles with rings~~ with two or three rings, usually buckle with the sheet. Due to its recent occurrence, this type of structure is never been satisfactorily studied & a method of analysis is still to be developed. J. L. Taylor & N. J. Hoff (13) made some calculation in this direction, but there are a number of their assumptions which seems doubtful. ~~to the present reports~~

~~However, before the discussion such complicated structure, it seems essential to understand the fundamental physical principle of the buckling phenomenon, and to study of the elementary components of the structure. The following is such an attempt.~~

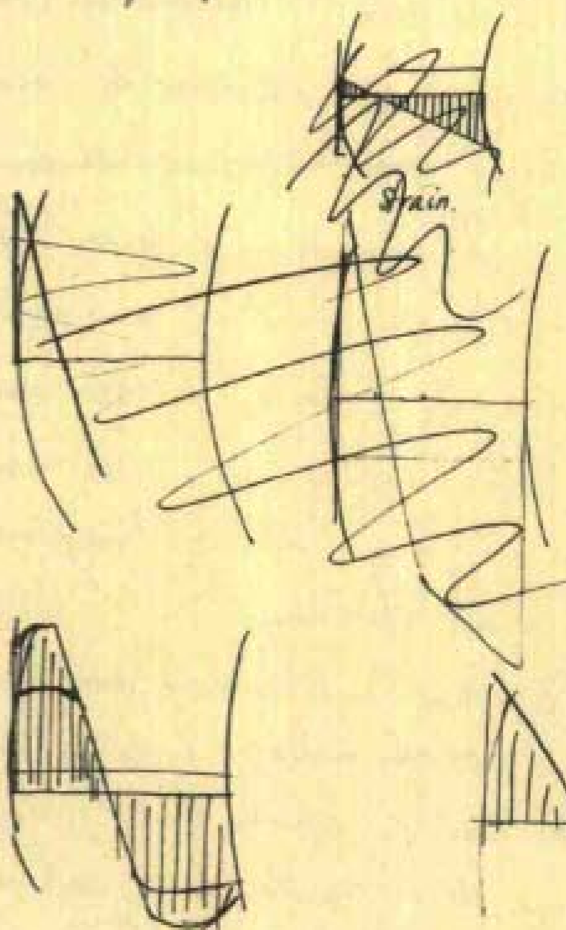
However, before the discussion of such complicated structure, it is useful to have a clear understanding of the buckling & related phenomena. In the literature of thin shell structure, it is frequent that the writers do not emphasize the difference between buckling ~~stress~~ load and maximum load. It is ~~the point of~~ evident that the ~~linear~~ small deflection theory can

7)

at most give the elastic buckling load of the structure. The condition beyond the buckling load is very complicated. If the buckling load is not far [i.e., the load is not above width load] low and the waves rather shallow, then it can be expected that there is no place in the sheet where the elastic limit is exceeded. Then we can calculate the relation between load and deflection by using large deflection theory and Hook's Law. The assumption of large deflection and consequently the resulting non-linear differential equation is necessary for the reason that even in case of rather shallow wave, the maximum deflection in normal direction may amount to 100 times the sheet thickness.

But if the deflection involves very small radius of curvature or the initial buckling load is high [as is the case for curved sheet], then there must be some place in the sheet where the yielding point is exceeded. Hence not only that we have to take into acount the large deflection but also the plastic flow of the material. But here the Kármán's well-known theory of

plastic buckling with two elastic modulus is not sufficient. → Because not only the plastic region ~~is entirely~~ localized, but the bending stress ~~usually~~ usually far exceeds the direct ~~compressive~~ stress & so the correct diagram of stress distribution is that of Fig. 1a instead of Fig. 1b.]]



]] T. Pöschl (23) recently has ~~attempted~~ developed the theory of column for stress distribution like that shown in Fig. 1c. Therefore for the investigation of plastic failure of plates & shells ~~are~~, a further attention to the case shown in Fig. 1a is necessary.

Fig. 1a

Fig. b: for rather thick plates & shells

~~All~~ The theory of plastic buckling with two elastic modulus is developed by F. W. Gockel (18) & ~~W. Kaufmann~~ (19)(20)(21). However, in their calculation, ~~we~~ the Navier's hypothesis is

retained. But due to the fact that for all ordinary materials (9) only thick plates & shells fail by plastic buckling, it seems necessary to combine it with R. V. Southwell's more general theory (1) to give a satisfactory solution.

From the above consideration, it is quite evident that the load after buckling is a complicated matter. The question whether the load will increase or decrease depends upon the buckling load and characteristics of the shell. If the load still increases after buckling, then the maximum load will be higher than the buckling load. But if the load do not increase then the buckling load is the maximum load. In the first case, the specimen fails by yielding. In the second case the specimens fail by instability. This shows the importance of differentiation of buckling load and maximum load.

(II.)

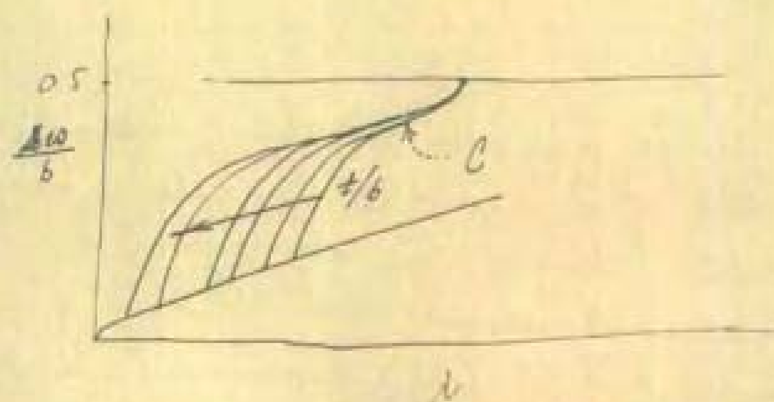
Plate

The failing of a flat plate of ordinary dimension is clearly of the first kind, i.e., plastic yielding. The plate buckles at load near to the theoretical buckling load calculated from the theory of small deflection. & The load increases with the deflection & the wave pattern becomes more & more complicated

(18)

with appearance of small waves at the supported edges.

K. Marguerre, & E. Trefftz and A. Kromm (14), (15), (16) applied the theory of large deflection to investigate the load & deflection relation after buckling. And they were the first to obtain a ~~theoretical~~ calculated result which agrees with ~~the~~ numerous experiments. However, they assumed the Hooke's stress-strain relation, & so the theory can only expected to work at load few times the buckling load when no plastic region appears. There is another difficulty which is due to the energy ~~method~~ used by them, i.e., the necessity of assuming different wave pattern for different loading & different l/b ratios. This is inferred from the experimental results of Dunn, as shown in Fig. 2.



K. Marguerre only gives one curve C independent (11)
of t/b, probably because he assumed only one type of wave form.

(III)

Cylindrical Shell

The failing of cylindrical shell is of the second type, i.e., fails by instability, and the buckling load & the maximum load coincides. The notable results of experiments are [see fig. 3]

(1) the large scatter of experimental points

(2) the very low value of buckling stress as compared to the theoretical result.

(3) the tendency towards lower stress at larger R/t ratio.

The theoretical study of instability of thin cylinder assuming the small deflection theory was carried out by a number of writers and was very well summarized by K. v. Sanden, F. Tölke (3) and W. Flügge (4). W. Flügge in the later part of his paper tried to explain the discrepancy by considering the end conditions more carefully, that is, consider the end to be supported in grooves which does not allow displacement but is free to change of slope. But by so doing he obtained a set of inhomogeneous boundary conditions, and therefore like an eccentrically load column, it

~~Can only fail by yielding~~. gives no instability problem (2)
Instead the waves gradually grows deeper and deeper & fail by yielding. But ~~all~~ the experiments ~~made~~ shows that most of the cylinders fail by sudden buckling accompanied by the emission of noise. this is typically an instability phenomena. ~~which can hardly be explained~~ W. Flügge and later L.H. Donnell (7) also introduced the idea of initial ~~defle~~ imperfection of the specimen. But in a series of tests made by W. L. Holland, it is impossible to detect any initial imperfection by naked eye.

L. H. Donnell (7) also ~~introduced~~ ~~used the~~ ~~expe~~ extended the theory to large deflection and set up the failing criterion by yielding. this is also questionable ~~due to~~ because for at least the test made by L. Holland in very thin steel cylinders, the buckling wave can be removed completely by unloading just after buckling occurs.

It seem at present, a critical examination of the theory of thin shells, originally developed by A.E.H. Love is necessary. ~~the~~ Although the Navier assumption could be retained, it is probably necessary to use theory of finite deflection & examine the instability under finite deflection. [See Fig 4] this is suggested

by the experimental fact that the discrepancy between classical theory and experiment is larger ~~for the case of~~ when R/t is larger which means that the ratio of deflection to thickness is larger. It is 13)



Fig. 4.

also usually at this stage of theory of thin shell, to investigate experimentally the buckling phenomena in more detail. An accurate measurement of initial imperfection, and wave patterns, and a high speed moving picture of buckling process are very desirable.

(II)

Curved Sheet

The curved sheet can be considered as an intermediate between the flat plate and the cylindrical shell. Therefore the way it fails can be either by yielding like plate or by instability as cylinder. When the ratio b/R is small, then it fails like plate; when b/R is large, it fails by instability. This is clearly shown

by the rather meager experimental results of W. A. Wenzek [29] [Fig. 5] (14)

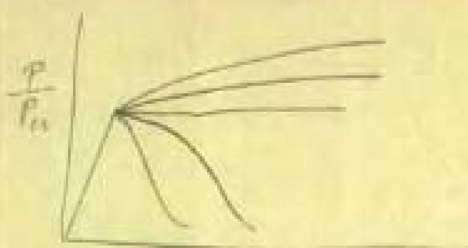


Fig. 5

$\frac{\sigma}{\sigma_{cr}} \sim \varepsilon$

Due to this difference in the character of failing, the well-known method developed by E. E. Sechler, although an excellent first approximation, needs improvement especially for large curvatures. However, a more satisfactory treatment of this problem probably have to await the solution of apparent mystery of thin cylindrical shell.

It is interesting to notice that the ratio of experimental & theoretical buckling loads is higher for smaller t/R and lower for larger t/R.

(V)

Stiffened Plates & Shells

The usual practice in the calculation of stiffened plates & shells is to distribute the stiffeners uniformly over the shell, and take the sheet as having different elastic properties in direction ⊥ to stiffeners & parallel to stiffeners, the so-called orthotropic shell & plate. But this practice is only good for shells or plates stiffened by corrugation. In case of stiffeners placed not very

close together, then this is only a first approximation. (15)

J. L. Taylor (10) even went so far as to assume the rings also uniformly distributed. This is very doubtful from the point of view of actual construction.

¶ R. Barbré (17) made a very elaborate calculation of the buckling load of the stiffened plate using the exact theory of small deflection. His results agree with that of the well-known work of S. Timoshenko who used the much simpler energy method. However, the calculation of load beyond buckling, or the maximum load remains to be done.

N. J. Hoff (13) applied the energy method to solve the problem of buckling of stiffened cylinders with rings buckle with the sheet. His work can be said as pioneering in this field, but it suffers the defects of a rather arbitrary wave pattern, neglection of effect of sheet and small deflection theory. ~~However, in~~

(VI) (16)

Mathematical Methods

The methods used in the study of thin plates & shells fall into two general headings:

(1) Method of Equilibrium — This is to find a deformed shape of the shell or plate other than straight compression, such that it will be in equilibrium with the external load. When the boundary conditions are homogeneous, ~~the~~ the problem is reduced to find the characteristic values of the differential equation of deflection. ~~The~~

(2) Method of Lowest Energy, or Energy method of Timoshenko. In this method, the difference between the strain energy and the potential energy of applied load is made to be a minimum. The problem is essentially a variation problem. The most popular method to solve it is that due to Rayleigh & Ritz.

For ~~most cases~~ almost all cases, the second method involves less thought. ~~and~~ In the case of non-linear equations, like that obtained by means of finite deflection theory, or ~~stiffened shell, the calculation~~ the energy method is almost the only means to solve ~~the equations~~. The uniqueness of solution by

using energy method for large deflection is studied by K. Marguerre (12) (17)
The reason why this is necessary is this: the energy expression in case of small deflection theory is a positive definite quadratic form, so that condition of a extremum is a linear equation. Therefore there is only one solution. The solution is also a minimum, because the quadratic form is positive definite. This is the well-known Kirchhoff's principle of uniqueness of solution. But the energy expression in case of large deflection theory involves third, and higher order terms, the extremum condition is not linear. Therefore there are more than one solution, for each load condition. To decide which is the true minimum, one has to use the second variation. This is done for the case of column by K. Marguerre (2) using Ritz' method.

Another difficulty of using energy method for large deflection theory is to assume proper deflection forms, especially the tangential component. The normal component is fairly easily obtained from experiments. Both L.H. Donnell (7) & K. Marguerre (14) (15) (16) used the equilibrium differential equation

(18)

combined with the assumed wave form of normal component to obtain proper wave form of tangential components. Then the tangent components are expressed in terms of parameters & functions of assumed normal component. These values are then put into the energy integral and minimized to determine the parameters.

The equilibrium differential equations used by K. Marguerre (15) and L.H. Donnell (7) have to be critically examined too. The reason is that they used the equilibrium differential equation for small deflection which is no longer true in the theory of large deflection; the point is that certain second order terms of deflection are neglected. This is evidently incorrect, because the inclusion of second order terms in the calculation is the main feature of large deflection theory.

Note: The reviewer has no wish of claiming all the opinions expressed in this report to be his own. Some of them are really first suggested by Mr. W.L. Holland and Dr. E.E. Sechler.

1.2.2
The Buckling of Spherical Shells by External Pressure
球壳外压屈曲

这是作者发表于 1940 年的“The Buckling of Spherical Shells by External Pressure”（球壳外压屈曲）一文的原始手稿，共有 26 页。稿中用比较花的草体书写的部分是 Theodore von Kármán（冯·卡门）的增改。

在 20 世纪 30 年代，特别是在 1930—1937 年这段时期，飞机工业正在设计和生产具有全金属壳体结构的新型飞机。这种结构具有重量轻而强度高的优点。但当其受到的载荷超过某一数值，壳体发生皱瘪而失效，这种现象称之为屈曲。如果采用经典的线性理论作计算，发生屈曲的临界载荷值比实验值要大许多。飞机设计师为了安全起见，只能根据相当分散的试验数据来确定临界载荷的数值。

作者在系统地分析了前人的理论和实验工作以后，认为既然经典的线性理论能精确地预报平板的屈曲载荷以及屈曲后的状态变化，很可能在平板和具有弯曲形状的壳体发生屈曲时，这两种物理过程间存在着一个尚未认识到的本质区别。对于有初始曲率的球壳，作者认为 von Kármán 推导的非线性方程反映了这种区别，因此以这个方程为出发点。

作者为了解决有关临界载荷的矛盾，并且回答究竟多大载荷能使受压球壳屈曲失效的问题，在这篇论文中提出了有关具有弯曲形状的壳体发生屈曲的机制的新观点。作者认为：经典理论之所以失败，在于没有考虑到，在加载过程中球壳除了保持球形位形以外，还可能存在多个位能更低的其他位形。壳体在受到外界干扰时，会被激发而从球形位形跃变到位能较低的某个位形上去。因而作者提出：有必要区分经典线性理论所给出的“上”屈曲载荷以及使壳体发生有限变形的屈曲时的“下”屈曲载荷。前者可以在试验中小心避免不对称等初始缺陷而达到，而在设计中所采用的临界载荷只能是后者。

由于作者的计算结果确实和试验值很接近，上述理论很快被学术界和工程界所接受。

1)

The Buckling of Spherical Shells by External Pressure

Th. von Kármán and Hsue-shen Tsien

Introduction General considerations.

The general theory of thin shells was developed by A.E.H. Love. He assumed that the deflection from the unloaded position is small, and for that reason he neglected all terms higher than the quadratic in the energy expression and obtained three linear differential equations for the determination of the new equilibrium position of the shell under given forces. The theory of buckling of thin shells is based essentially on Love's equations. The buckling of cylindrical shells of uniform thickness under the action of uniform distributed axial load was calculated by R. Lorenz, R.V. Southwell, S. Timoshenko, W. Flügge and L.H. Donnell, and others. The same problem was also investigated experimentally by many authors, especially by E.E. Lundquist and L.H. Donnell. Unfortunately a systematic discrepancy was found between the theoretically calculated buckling loads and experimentally obtained; the theoretical

(2)

(are as much as)
value ~~is~~ 3 to 4 times higher than the experimental values.

~~This failure of theory must be very disheartening to all who work in the field of applied mechanics.~~ To remedy this situation

W. Flügge first considered the deviation of assumed end conditions

of the cylindrical shell from that realized in the laboratory.

However, ~~this attempt is not successful~~ (this end effect is not sufficient to explain the discrepancy), because ~~the end effect~~ (it affects only a)

~~it is only limited to a~~ region of about the length $\sqrt{Rt}$ (near) ~~from~~ the (supports)

~~ends~~, where R is the radius of the shell and t the thickness.

The cylinders tested, however, usually have a length much larger

than twice this value. Furthermore, (Flügge's) ~~his~~ analysis (would) indicate~~s~~

a progressive increase of the wave amplitude until plastic

deformation sets in, (whereas the) ~~However,~~ experimental evidence indicates

that the failure of ~~thin~~ cylindrical shells under compression

is not progressive but ~~is~~ very rapid.

Another attempt was made by W. Flügge and later by L. H.

Donnel ~~who tried~~ to lower the theoretical buckling load by

(taking into account the) ~~introducing~~ initial deviation of the form of (the) shell from the

true cylindrical shape, ~~and then determine~~ the buckling (load) ~~load~~,

or rather the failing load by (a) plastic failure of the material.

(would then determined)

22

(2a)

several

This explanation has two drawbacks: Firstly, the necessary

obtain

deviation of the shape of cylinder to make the calculated buckling

the low values of the buckling load found

values same as those experimentally found is usually more than

one has to assume deviations from the cylindrical shape

10 (continued on page 3)

as large as ten — page 3.

(2b)

exact

Furthermore, the intial deviations from the true cylindrical

would cause the to increase

form will make the process of deformation a gradual one. gradually

p3

which

This is again contrary to laboratory experience.

(13)

times the shell thickness. Such a large deviation from the shape of cylinder, the phenomenon would be easily observed by the eye. This is not substantiated by experience. Secondly, the failure of a cylindrical shell is not necessarily plastic failure (yielding), especially when the wall of the cylinder is very thin. In many cases, it was observed that upon removing the load the buckling waves completely disappeared. Therefore, the phenomenon must be completely elastic, instead of being plastic as assumed in L. H. Donnell's analysis I (Ref. 16) (discrepancy between theory and experiment).

A similar situation exists in the case of the buckling of spherical shells under uniform external pressure. The theoretical buckling load based upon Love's equations has been calculated by R. Zoelly, E. Schwerin, and J. A. van der Neut.] While no systematic experimental work has been done on this problem, at least to the author's knowledge, some tests made at GALCIT by

] If the buckling stress σ_c is defined by

$$\sigma_c = \frac{p_c R}{2t}$$

where p_c is the buckling pressure, t the thickness, & R the

radius of the sphere, then the theory gives

$$\sigma_c = \frac{1}{\sqrt{3(1-\nu^2)}}\, E\left(\frac{t}{R}\right)$$

where ν is Poisson's ratio and E Young's modulus of elasticity.
E. E. Sechler & W. Bollay at the California Institute of Technology indicated that the experimental buckling load is only about 1/4 that of the theoretical value.

Besides these differences between the theoretically and experimentally obtained buckling loads, the wave form predicted by the theory is also at variance with the laboratory experience. According to theoretical calculation the same load would produce buckling either inward or outward; the experiments show that the shell has a definite preference to buckle inward. For the case of a spherical shell it is observed that the buckling wave is restricted to a small dimple subtended by a solid angle of about 15°. The theory, however, predicts a wave form extended over the whole spherical surface.

What may be the reason for these discrepancies between the theory and the experimental evidence?

between the physical process of the buckling of a flat plate and a curved shell, that is not embraced by the previous theory.

It is advantageous to start the investigation with the case of a spherical shell loaded by a uniform external pressure because of the simplicity of the problem resulting from the symmetry of the shell. Consider a segment of an extremely thin spherical shell with edges as shown in Fig. 1. If the shell is sufficiently thin, the bending stiffness, which is proportional to t^3, can be neglected, hence the bending strain energy can be completely neglected. Under this assumption, the strain energy of the shell (i.e. the energy due to extension or compression of the center surface and [illegible]) is equal to zero in the deflected position (3) as it is in the undeflected position (1), Fig. 1. In other words, neglecting the bending stiffness, the shell will be in equilibrium in the deflected position (3) without the aid of any external pressure applied to the shell surface.

On the other hand, the intermediate positions between (1) & (3) do involve compression of the shell elements, & therefore, the shell can be in equilibrium in these positions only here

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(4a)

I ~~There~~ It is very ~~small probability~~ unlikely ~~of defect in~~ that there is an error in the fundamental ~~disipline~~ equations of the theory of elasticity. ~~Because~~ For example in the ~~of~~ case of a flat sheet ~~under end compression~~, not only the buckling load but also the behavior of the sheet after buckling is predicted accurately by the theory. ~~This opinion is shared by H.L. Cox in his recent lecture before the Royal Aeronautical Society.~~

Hence there must be an essential difference

II In this paper a new conception of the mechanism of the collapse of curved sheets is presented.

(6)

by
~~with the aid of an~~ external pressure. When the deflection of the shell go beyond the the position (2) of above position (3), a negative external pressure is necessary to maintain equilibrium as the compressed elements are trying to force the shell to take up the equilibrium position (3). The pressure-deflection curve, under the assumption of negligible bending stiffness is, therefore, of the form shown in Fig. 2a.

The effect of the bending stiffness is to increase the positive external pressure necessary to hold the shell in equilibrium. We specify on this case clamped edge. ~~Thus~~ Then for increasing values of the bending stiffness of the shell, the pressure-deflection curve will take the form ~~as of shown~~ in Fig. 2b, shown by the curves 1, 2, etc.

Now the question arises: ~~what is the probable buckling load that will be observed in the laboratory?~~ at which time will the shell subjected to external pressure actually ~~[illegible]~~ fail? ~~In view of the fact that~~ The process equilibrium positions represented by the portions ~~of the curve~~ A-B (Fig. 2b) AB, A₁B₁, etc. ~~is~~ are highly unstable, ~~the hump in~~ hence if the load passes the peak A, it immediately drops to the value represented by B. Now ~~the curve~~ is ~~easily affected or decreased by~~ the critical ~~the~~ imperfections of shape, vibrations etc. the shape of the [illegible] the size of the initial [illegible] Furthermore, the curve in Fig. 2b is based upon the assumption ~~consideration~~ of symmetrical type of the deflection only, the peak in the

the peaks are relatively sensitive to

curve may be lowered by an anti-symmetric type of deflection, i.e. of one part of the shell moves out while the other part moves in. The analogous problem for a curved bar loaded by a concentrated force was recently investigated by K. Marguerre;+ he demonstrates that the process indicated in Fig. 7 — the "Durchschlag" according to the German terminology — is actually precipitated by an anti-symmetric type of deflection. Therefore it seems probable that — provided no especially careful measures are taken to keep [illegible] — the load observed in the laboratory corresponds to the minimum load B [illegible] of the curve in Fig. 2b instead of the point A. Thus the theoretical determination of the buckling load really consists of the finding of the minimum point in the load-deflection curve. This clearly brings out the defect in the old theory of thin shells. The old theory being a linear theory, is not able to give the non-linear load-deflection curve as shown in Fig. 2b. All it can do is only give the initial tangent AV to the curve,

+ Marguerre, [illegible] Stabilität [illegible]

Fig. 7a.

7a

It is believed that these simple consideration throw ~~a~~ some light on the problem of buckling of curved shells. Consider for example a complete spherical shell under the action of uniform external pressure. ~~and assume that a (certain) deflection subtended by the solid angle 2β. Then approximately~~ The classical theory is right in stating that until the buckling load obtained by the classical theory is reached, any infinitesimal deviation from the spherical form involves an increase of the potential energy of the shell and therefore the spherical form is stable. However the same classical theory fails to reveal that there are ~~portions~~ configurations not far away from the spherical form which involve a lower level of the potential energy and therefore the shell will actually jump over into these configurations. Such configurations ~~can be~~ are clearly indicated by the foregoing considerations; assume for example that a segment of the shell ~~remain~~ subtended by a solid angle 2β is deflected and takes the shape corresponding to the load minimum B, whereas the rest of the shell remains spherical. If it can be shown that the load corresponding to B is lower than the classical buckling load, the discrepancy

(shape of portion of the shell)

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between the failing load predicted by the classical theory and the found by actual experiments is explained. The problem is reduced to the determination of the value of the folded angle 2β which warrants the smallest value of the minimum load p_B.+)

This problem is attacked in an approximative way in this paper for the spherical shell. The analogous problem for the cylindrical shell under axial compression has been attacked by the author but the investigations did not yet reach a presentable state.

the reaction moment in the clamped edge must be taken by the rest of the shell. However the exact equilibrium configuration must be connected with a value of the potential energy which is even smaller than the value given by the approximated method. Hence the actual failing load is expected to be slightly lower than the calculated value.

+) The configuration indicated above is not an equilibrium position of the shell, since the curvature changes discontinuously at the boundary of the deflected region; in other words, the moment is not taken into account, that

(1)

for a spherical segment under without pressure

The Energy Expression and Equation of Equilibrium

The integration of the exact equations of equilibrium including non-linear terms is very complicated

To calculate the load-deflection curve of a spherical segment under uniform external pressure with inclusion of higher than quadratic terms in the energy of the shell in its equation of equilibrium is necessary, instead of the linearized equations in Love's theory. However a general consideration would be too complicated, therefore the following set of simplifying assumptions were made:

(1) The solid angle of segment is small.

(1) The buckling is restricted to a small portion of the shell and the edge of this region are clamped, however, the size of the this region is not restricted, & is to so determined as to make the buckling load a minimum.

2. (2) The deflection is rotationally symmetric

3. (3) The deflection of any element of the shell is parallel to the axis of symmetry

4. (4) The effect of lateral contraction is neglected, i.e., Poisson's ratio is assumed to be zero.

Assumptions (1) & (3) will give higher buckling value than it would be if these restrictions are removed.

Assumption (2) will not effect the buckling load because

Footnote * The "Durchschlag" of a spherical shell with small curvature under action of a concentrated force was calculated by C. B. Biezeno

Assumption 3 increases the buckling load, since it is on the nature of the variational method used in the following calculation. Assumption that any deviation from the exact buckling shape increases the buckling load (certainty) (19)

(1) probably will not change the results to any appreciable amount. The assumption was made in this paper only since it materially reduces the numerical work.

Fig. 3 indicates the notation used in the following calculation. The choice of the inclination θ of the meridian line as unknown parameter is essential for the simplicity of the equations.

As can be seen from figure 3, stretching or compression of the shell due to the assumed vertical deflection of the shell occurs only in the meridian direction. The element whose initial length along the meridian is equal to $dr/\cos\alpha$, has the length $dr/\cos\theta$ after deflection.

The strain is, therefore

$$\varepsilon = \frac{\frac{dr}{\cos\theta} - \frac{dr}{\cos\alpha}}{\frac{dr}{\cos\alpha}} = \frac{\cos\alpha}{\cos\theta} - 1$$

Hence, the strain energy due to the extension of the elements of the shell is given by

$$W_1 = \frac{Et}{2}\int_0^{\beta}\left(\frac{\cos\alpha}{\cos\theta} - 1\right)^2 2\pi r \frac{dr}{\cos\alpha}$$

or with $r = R\sin\alpha$ and $dr = R\cos\alpha\, d\alpha$

$$= \frac{ER^3}{2}\left(\frac{t}{R}\right) 2\pi \int_0^{\beta}\left(\frac{\cos\alpha}{\cos\theta} - 1\right)^2 \sin\alpha\, d\alpha \qquad (1)$$

TP

(10)

The two components of the curvatures of the shell at an arbitrary point P before deflection started are equal to $\frac{1}{R}$. After deflection, the curvature on the meridian plane is equal to

$$\frac{d\theta}{ds} = \frac{\frac{d\theta}{d\alpha}}{\frac{ds}{d\alpha}} = \frac{\frac{d\theta}{d\alpha}}{\frac{R\cos\alpha}{\cos\theta}}$$

Hence the change in curvature of a meridian section is

$$\frac{\frac{d\theta}{d\alpha}}{\frac{R\cos\alpha}{\cos\theta}} - \frac{1}{R} = \frac{1}{R}\left[\frac{\cos\theta}{\cos\alpha}\frac{d\theta}{d\alpha} - 1\right]$$

Similarly, the change in curvature of a section orthogonal to the meridian plane is

$$\frac{1}{R}\left[\frac{\sin\theta}{\sin\alpha} - 1\right]$$

The strain energy due to bending, therefore,

$$W_2 = \frac{Et^3}{24}\int_0^\beta 2\pi R^2 \sin\alpha \, d\alpha \, \frac{1}{R^2}\left[\left(\frac{\cos\theta}{\cos\alpha}\frac{d\theta}{d\alpha} - 1\right)^2 + \left(\frac{\sin\theta}{\sin\alpha} - 1\right)^2\right]$$

$$= \frac{ER^3}{2}\left(\frac{t}{R}\right)^3 \frac{2\pi}{12}\int_0^\beta \sin\alpha\left[\left(\frac{\cos\theta}{\cos\alpha}\frac{d\theta}{d\alpha} - 1\right)^2 + \left(\frac{\sin\theta}{\sin\alpha} - 1\right)^2\right] d\alpha \qquad (12)$$

The potential energy corresponding to the work done by the external pressure p is equal to the pressure p times the volume included between the initial and deflected surface of the shell: The volume enclosed between the initial surface and the plane of its circular edge is equal to

$$\int_0^a 2\pi r z \, dr = \left[2\pi \frac{r^2}{2} z\right]_0^a - \int_0^a \pi r^2 \frac{dz}{dr} dr$$

The volume enclosed by the deflected shell and the plane is equal to $R^3\pi\int_0^\beta \sin^2\alpha \, d\alpha$. Hence the potential energy is given by $W_3 = R^3\pi\int_0^\beta \sin^2\alpha(\tan\theta - \tan\alpha)\cos\alpha \, d\alpha$ (3)

$$= \pi\int_0^\beta R^2\sin^2\alpha \tan\theta \, R\cos\alpha \, d\alpha = R^3\pi\int_0^\beta \sin^2\alpha \tan\theta\cos\alpha \, d\alpha$$

The total potential energy, W, of the system is the sum of the strain energy and the potential energy due to external pressure. Thus, from equations (1), (2), (3)

$$\frac{W}{R^3\pi} = E\left(\frac{t}{R}\right)\int_0^\beta\left(\frac{\cos\alpha}{\cos\theta}-1\right)^2\sin\alpha\,d\alpha + \frac{E\left(\frac{t}{R}\right)^3}{12}\int_0^\beta\left[\left(\frac{\cos\theta}{\cos\alpha}\frac{d\theta}{d\alpha}-1\right)^2+\left(\frac{\sin\theta}{\sin\alpha}-1\right)^2\right]\sin\alpha\,d\alpha$$

$$+ p\int_0^\beta \sin^2\alpha\cos\alpha\,(\tan\theta - \tan\alpha)\,d\alpha \qquad (4)$$

At the equilibrium position, the total energy must be a minimum, therefore, the equation of equilibrium can be obtained by finding the relation between θ and α which will make the integral (4) a minimum. Using the rules of the calculus of variations, the following equation is obtained:

$$2E\left(\frac{t}{R}\right)\left[\frac{\sin\alpha\cos\alpha}{\cos\theta}\tan\theta\left(\frac{\cos\alpha}{\cos\theta}-1\right)\right] + \frac{E\left(\frac{t}{R}\right)^3}{6}\left[\cos\theta\left(\frac{\sin\theta}{\sin\alpha}+\tan^2\alpha\right) - \frac{\cos^2\theta}{\cos\alpha}\left(2\tan^2\alpha+1\right)\frac{d\theta}{d\alpha}\right.$$

$$\left. + \frac{\sin\theta\cos\theta\tan\alpha}{\cos\alpha}\left(\frac{d\theta}{d\alpha}\right)^2 - \frac{\cos^2\theta\tan\alpha}{\cos\alpha}\frac{d^2\theta}{d\alpha^2}\right] + p\sin^2\alpha\cos\alpha\sec^2\theta = 0 \qquad (5)$$

together with the boundary condition

$$\theta = 0 \quad \text{at} \quad \alpha = 0 \qquad (5a)$$
$$\theta = \beta \quad \text{at} \quad \alpha = \beta$$

(12)

Both equations (4) & (5) are unwieldy to handle. However, a great simplification results if β, the solid angle of the shell segment is small. This is confirmed by experiments. Then by expanding the sine & cosine functions into power series and neglecting terms higher than of the third order in α, θ and the derivatives of θ, equations (4) & (5) are reduced to:

$$\frac{W}{R^3\pi} = \frac{E\left(\frac{t}{R}\right)}{4}\int_0^\beta (\theta^2-\alpha^2)^2\alpha\, d\alpha + \frac{E\left(\frac{t}{R}\right)^3}{12}\int_0^\beta\left[\left(\frac{d\theta}{d\alpha}-1\right)^2+\left(\frac{\theta}{\alpha}-1\right)^2\right]\alpha\, d\alpha + p\int_0^\beta \alpha^2\theta\, d\alpha \qquad (6)$$

and

$$\alpha\frac{d^2\theta}{d\alpha^2} + \frac{d\theta}{d\alpha} - \frac{\theta}{\alpha} = \frac{6}{\left(\frac{t}{R}\right)^2}\alpha\theta(\theta^2-\alpha^2) + \frac{6p}{E\left(\frac{t}{R}\right)^3}\alpha^2 \qquad (7)$$

These are the simplified energy expression and the equation of equilibrium respectively. It is seen that the left side of (7) (terms on the) are linear in θ and in its derivatives.

In order to calculate the maximum deflection δ at the center, the ordinate z_0 (Fig. 3) at the center has to be first computed. By means of the boundary condition that $z=0$ at $\alpha=\beta$, the following relation is obtained:

$$z_0 + \int_0^\beta \frac{dz}{d\alpha}\, d\alpha = 0$$

→ They are included in the usual theory; the first term on the right side arises in the consequence of finite deformations.

(13)

Or $\qquad z_0 = R\int_0^\beta \tan\theta \cos\alpha \, d\alpha \qquad (8)$

Before deformation, the ordinate at the center is equal to $R(1-\cos\beta)$

Therefore the deflection δ at the center ~~can be calculated as~~ is given by

$$\delta = R\left\{(1-\cos\beta) - \int_0^\beta \tan\theta \cos\alpha \, d\alpha\right\} \qquad (9)$$

If β is again assumed to be small, equation (9) is simplified to

$$\delta = R\left\{\frac{\beta^2}{2} - \int_0^\beta \theta \, d\alpha\right\} \qquad (10)$$

Approximate Solution by ~~Energy~~ the Rayleigh-Ritz Method

To calculate the load-deflection curve, one can either solve the differential equation (7) or minimize the integral (6) directly by means of the Rayleigh-Ritz method. Due to the non-linear character of the differential equation (7), it is difficult, if not impossible to solve it analytically. Therefore ~~the~~ in this paper the Rayleigh-Ritz method is used. To apply ~~In~~ this method, at first a plausible form of the deflection satisfying the boundary conditions has to be found ~~first~~. ~~From the symmetry of deflection,~~

It is easily seen that for reasons of symmetry θ must be an odd function of α. The simplest assumption for a function $\theta(\alpha)$ which satisfies the boundary conditions is

$$\theta = \alpha\left[1 - \kappa\left(1 - \frac{\alpha^2}{\beta^2}\right)\right] \qquad (11)$$

where κ is an undetermined parameter. Substituting the expression (11) in equation (10) we find that κ is connected with the deflection δ by the relation

$$\kappa = \frac{4\delta}{R\beta^3} \qquad (12)$$

If we introduce (11) in the energy expression (9) and carry out the integrations, we obtain

$$\frac{W}{R^3\pi} = \frac{Et}{60R}\beta^6\left(\kappa^2 - \frac{\kappa^3}{2} + \frac{\kappa^4}{14}\right) + \frac{Et^3}{18R^3}\beta^2\kappa^2 - \frac{p\beta^4}{4}\frac{\kappa}{3} \qquad (13)$$

The equilibrium condition between the pressure p and the deflection δ is obtained by putting $\frac{\partial W}{\partial \kappa} = 0$

$$\frac{1}{R^3\pi}\frac{\partial W}{\partial \kappa} = \frac{Et}{60R}\beta^6\left(2\kappa - \frac{3\kappa^2}{2} + \frac{2\kappa^3}{7}\right) + \frac{Et^3}{9R^3}\beta^2\kappa - \frac{p\beta^4}{12} = 0 \qquad (14)$$

Introducing $\sigma = \frac{pR}{2t}$, where σ is the uniform compression stress produced by the pressure p under the assumption of small deflections, the following relation is obtained from Eq. (14)

$$\frac{\sigma}{E} = \frac{\beta^2}{5}\left(\kappa - \frac{3}{4}\kappa^2 + \frac{1}{7}\kappa^3\right) + \frac{2}{3}\frac{t^2}{R^2}\frac{\kappa}{\beta^2} \qquad (15)$$

Introducing the deflection δ and using Eq. (12), then Eq. (15) can be rewritten as

$$\frac{\sigma}{E} = \frac{4}{5}\left(\frac{\delta}{R} - 3\frac{\delta^2}{R^2\beta^2} + \frac{16}{7}\frac{\delta^3}{R^3\beta^4}\right) + \frac{8}{3}\frac{t^2\delta}{R^3\beta^4} \qquad (16)$$

If σ/E is plotted as function of δ/R, it is found that the curve for load vs deflection has the shape indicated in Fig. 2, when $\beta > \sqrt[4]{\frac{32}{3}}\sqrt{\frac{t}{R}}$ or

$$\frac{t}{R} < \frac{1}{4}\sqrt{\frac{3}{2}}\,\beta^2 \qquad (17)$$

If t/R is larger than the value on the right side of (17) the load increases with the deflection without having a peak and a minimum; for $t/R = \frac{1}{4}\sqrt{\frac{3}{2}}\,\beta^2$ the curve for load vs deflection has an inflexion point with horizontal tangent.

To find the smallest of the minimum values of the function $\sigma/E = f(\delta/R)$, first vary β, and determine $(\sigma/E)_{min}$ for given values of δ/R. By differentiation of (16) with respect to β^2 the value of β^2, which makes σ/E to a minimum is obtained as

$$\beta^2 = \frac{32}{21}\frac{\delta}{R} + \frac{20}{9}\frac{t^2}{\delta R} \qquad (18)$$

and substituting this value of β^2 into Eq. 16, the following relation is obtained

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$$\sigma/E = \frac{4}{5}\frac{\delta}{R}\,\frac{1+\frac{3}{280}\left(\frac{\delta}{t}\right)^2}{1+\frac{24}{35}\left(\frac{\delta}{t}\right)^2} \qquad (19)$$

or

$$\frac{\sigma R}{Et} = \frac{4}{5}\frac{\delta}{t}\,\frac{1+\frac{3}{280}\left(\frac{\delta}{t}\right)^2}{1+\frac{24}{35}\left(\frac{\delta}{t}\right)^2} \qquad (20)$$

It is seen that $\frac{\sigma R}{Et}$ is a function of δ/t only. The function defined by Eq. (20) is plotted on Fig. 4. The physical meaning of this curve is the following: for values $\frac{\sigma R}{Et}$ and δ/t corresponding to points that lie below the curve, no equilibrium position is possible, whatever values the solid angle of the segment 2β may take. (16a) Hence the maximum of the curve, which corresponds to $\delta/t =$ and $\sigma = 0.4908\, E\, t/R$ gives the smallest value for any load through which the load has to pass before the collapse of the shell, provided the latter is of exact spherical shape and its deformation occurs with exact axial symmetry. It is noteworthy that the maximum corresponds to a deflection: thickness ratio as small as $\delta/t = 1.2479$. The minimum value of the load that is able to keep the shell in a deflected shape and corresponds to $\delta/t = 3.1481$ is given by $\frac{\sigma R}{Et} =$ or

$$\sigma = 0.2377\, E\, t/R. \qquad (21)$$

(16a)

It is be particularly clear when Eq. (16) is rewritten in the following form:

$$\frac{\sigma}{E}\frac{R}{t} = \frac{1}{105}\left(\frac{\delta}{t}\right)\left[21 - 63\left(\frac{\delta}{t}\right)\frac{\frac{t}{R}}{\beta^2} + \left(48\frac{\delta^2}{t^2}+70\right)\frac{\left(\frac{t}{R}\right)^2}{\beta^4}\right]$$

Using $\beta^2/\left(\frac{t}{R}\right)$ as a parameter, a family of curves expressing the relation between $\frac{\sigma R}{Et}$ & $\left(\frac{\delta}{t}\right)$ can be drawn. (Fig. 4) Then the relation given by Eq. (20) represents the envelope of this family of curves.

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Application to the buckling problem. Comparison with the experiment

The results of the last section indicate that equilibrium problems involving finite deflections exist at much smaller loads than the buckling load given by the classical linear theory. To apply our method to the problem of buckling of a spherical shell under external pressure we have to modify our expression (6) for the energy by including the strain produced by uniform compression of the spherical shell $\frac{p}{2}$ the pressure p before buckling. We obtain

$$\frac{W}{R^3\pi} = \frac{Et}{?}\int_0^{\beta}\left[\frac{1}{2}(\Theta^2-\alpha^2)-\frac{pR}{2Et}\right]^2\alpha\,d\alpha + \frac{Et^3}{12R^2}\int_0^{\beta}\left[\left(\frac{d(\Theta-\alpha)}{d\alpha}\right)^2+\left(\frac{\Theta-\alpha}{\alpha}\right)^2\right]\alpha\,d\alpha$$

$$+\,p\int_0^{\beta}\alpha(\Theta-\alpha)\,d\alpha \qquad (22)$$

Using again Eq. (11) for Θ the equilibrium condition $\frac{\partial W}{\partial k}=0$ leads in this case to the following relation:

$$\frac{pR}{2Et} = \frac{\sigma}{E} = \frac{1}{70}\beta^2\left[28-21k+4k^2\right] + \frac{4}{3}\left(\frac{t}{R}\right)^2\frac{1}{\beta^2} \qquad (23)$$

Substituting k from Eq. (12), we obtain

$$\frac{\sigma}{E} = \frac{2}{5}\beta^2 - \frac{6}{5}\frac{f}{R} + \left[\frac{32}{35}\frac{f^2}{R^2} + \frac{4}{3}\left(\frac{t}{R}\right)^2\right]\frac{1}{\beta^2} \qquad (24)$$

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$$\beta^2 = \sqrt{\frac{11}{7}\left(\frac{\delta}{R}\right)^2 + \frac{10}{3}\left(\frac{t}{R}\right)^2}$$

If we now determine the minimum of σ/E for a given value of δ/R by varying β^2 we obtain

and
$$\frac{\sigma}{E}\frac{R}{t} = \frac{4}{5}\left\{ \sqrt{\frac{10}{3} + \frac{16}{7}\frac{\delta^2}{t^2}} - \frac{3}{2}\frac{\delta}{t} \right\} \qquad (30)$$

This relation is plotted in Fig. 5. For $\delta/t \to 0$ we obtain

$$\sigma = 1.4606\, E\frac{t}{r} \qquad (31)$$

This value of the buckling stress of course is much higher than the value given by the linear theory. This is expected because the assumed buckling mode is far away from that resulting from the linear theory, i.e., it is a very "unfavorable" shape for infinitesimal deflections. The minimum of $\frac{\sigma R}{Et}$ is equal to 0.18258, i.e. the minimum load necessary to keep the shell in the buckled shape corresponds to

$$\sigma_{min} = 0.18258\, E\, t/r \qquad (32)$$

This shows that the assumed shape is "favorable" for finite deflections.

The value of β corresponding to σ_{min} is $\beta = 3.821\sqrt{\frac{t}{R}}$, the value corresponding to the value given by Eq. (31) is equal to $\beta = 1.6174\sqrt{\frac{t}{R}}$. The deflection corresponding to σ_{min} is equal to about

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16-times the thickness of of the shell.

E.E. Sechler and W. W. Bollay found by subjecting a thinwalled copper hemisphere of 18" radius to external fluid pressure

$$\sigma = 2480 \text{ lbs/inch}^2$$

With $E = 14.5 \times 10^6$ lbs/inch², $t = .020"$, i.e. $\frac{R}{t} = 900$ this value corresponds to

$$\sigma = 0.154 E \frac{t}{R} \qquad (28)$$

The experimental vessel compare quite favorably with the value (27) given by our approximate theory. The experiment gave $\beta = 8°$, $\delta \cong 0.25$, i.e. $\frac{\delta}{t} = 12.5$. It is seen that the theory reproduces with fair approximation the physical process. The linear theory gives $\sigma = 0.606 E\left(\frac{t}{R}\right)$; the value of β (corresponding to the first nodal line) would be according to the linear theory 3.3°

It appears that for the buckling of curved shells there are an "upper" buckling load given by the linear classical theory and a "lower" buckling load which equal to the minimum load necessary to keep the shell in a buckled shape involving finite deformations.* The essential progress of the theory is that it determines the lower buckling load independently of the initial imperfectness of the specimen or the

* In Fig. 5 the probable shape of the curve for load vs deflection is given connecting both values to indicated by dotted line.

load arrangement; whereas all previous attempts to take into account finite deformations lead to buckling loads which depend on arbitrary assumptions concerning the magnitude of such imperfections or lack of symmetry. It seems that the upper buckling load can be approached experimentally only by application of extreme precaution; the engineering test will invariably give values very near to the lower buckling load and of course this value is to be specified for design.

The new theory also reveals the essential difference between the buckling of flat plates and curved sheets. The finite deformations of buckled flat sheets were calculated by several authors, e.g. S. Timoshenko, H. L. Cox, H. Yamana, K. Kondo, and K. Marguerre. The results of these authors do not agree completely due to different simplifying assumptions introduced in the computations. However all investigators agree that after the buckling of the plate an increasing load is necessary to increase the deflections. The load falls only if the elastic limit is passed. The senior author has shown for the analogous case of straight beams that

21.

Due to the ~~falling load~~ decrease of the load ~~beyond it~~ after buckling ~~it is much more difficult to reach the theoretical buckling load~~ the ~~scatter~~ experimental scatter ~~of the~~ is ~~buckling load~~ much larger if the buckling occurs in the plastic range than if it occurs in the ~~purely~~ elastic range. ~~We have shown in this paper that the load after buckling~~ ~~The phenomenon of decrease~~ The (rapid) decrease of the load after buckling revealed in this paper ~~is a pure~~ for curved shells is a pure elastic phenomenon ~~and therefore changes necessarily~~ and therefore changes the entire aspect of the buckling problem.

(~~theoretical and formational~~)

~~important difference is that~~ therefore, that

1.2.3

The Buckling of Thin Cylindrical Shells under Axial Compression

柱壳轴压屈曲

这是作者发表于 1941 年的 “The Buckling of Thin Cylindrical Shells under Axial Compression”（柱壳轴压屈曲）一文的部分手稿和算草。作者反复推敲，前后写了 5 份文稿，这里选印每份文稿的前两页；为论文所做的演算草稿有 800 多页，这里选印其中的 18 页；再选印了存放手稿的信袋的正面。总共选印了 29 页。

作者在 1940 年发表的 “The Buckling of Spherical Shells by External Pressure”（球壳外压屈曲）一文中，已经提出了计算屈曲临界载荷的能量跃变准则。这里，作者将这一准则推广到应用更为广泛的柱壳的情况。

作者为了求取圆柱壳屈曲的临界载荷，围绕寻找壳体可能达到的位形，进行了大量推导和演算，手稿长达 800 多页。这里仅选用其中开头和最后的部分。开头部分选了 1—3 页及 117—121 页，这部分有圆柱壳非线性方程的推导，并对位移（u, v, w）的形式作了反复试探和计算，力图正确表达柱壳发生菱状皱折的形态。最后，作者在 733—736 页上终于找到了比较满意的有关位移 w 的形式，在 741 页上给出了特征方程，746 页上给出缩短量 ε 与端部应力 σ 的关系，750 页是关于特征值 λ 的数值计算，784—786 页是部分计算的表格。

请注意作者在存放手稿的信袋上写的 “Final”（最后的定稿）和 “Nothing is Final”（没有什么认识是最后的）的字样，其中含有深刻的哲理，说明了真理的相对性，科学家追求真理是永无止境的。

First Draft

1

THE BUCKLING OF thin CYLINDRICAL SHELLS UNDER AXIAL COMPRESSION

Th. von Karman and Hsue-shen Tsien
California Institute of Technology

In two previous papers (Ref.1 and Ref.2), the authors have discussed in detail the inadequacy of the classical theory of thin shells in explaining the buckling phenomenon of ~~both the~~ cylindrical ~~shells~~ and ~~the~~ spherical shells. It was shown that not only the calculated buckling load is 4 to 5 times higher than that experimentally observed, but the buckling wave pattern found is also different from that predicted. It was pointed out, furthermore, that the different explainations for this discrepancy advanced by L.H. Donnell (Ref. 3) and W.Flügge (Ref. 4) are untenable when certain conclusions drawn from these explainations are compared with the experimental facts. The authors are then led by ~~both~~ the theoretical investigation on spherical shells (Ref.1) ~~and~~ a model experiment (Ref.2) on thin columns ~~supported by~~ with non-linear elastic support to the belief that the buckling phenomenon of curved shells can only be explained by means of ~~the~~ non-linear large deflection theory. ~~The crucial point seems to be~~ It is found in the two cases investigated that the ~~rapid drop in~~ load necessary to keep the shell in equilibrium drops very rapidly with increase in wave amplitude ~~once the shell started to buckle~~. This characteristic of dropping load shows, first of all, that there is a release of the elastic energy stored in the shell once the buckling has started, and thus explains the observed rapidity

of the buckling process. Furthermore, this characteristic also brings in the possibility of a decrease in the buckling load when there are slight imperfections in the test specimen and when there are vibrations during the testing process.

In this paper, the authors will show by means of an approximate calculation the dropping load characteristic of a uniform, thin cylindrical shell under axial compression. Consequently, they hope that they have thus offered an aceptable explaination of the observed facts.

Stresses in the Median Surface and the Expression for the Total Energy of the System

If u, v, and w (fig. 1) are the displacement of a point on the median surface of the shell in the axial, x, direction, the circumferential, y, direction, and the radial direction, then the unit strains in the x and y directions, ε_x, ε_y and the unit shear γ_{xy} at a point in the median surface can be expressed in the following form, including terms up to second order:

$$\left.\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \varepsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - \frac{w}{R} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) \end{aligned}\right\} \quad (1)$$

where R is the radius of the undeformed median surface of the shell. The stresses and the strains in the median surface of the shell are, however, related to each other by the following expressions:

THE BUCKLING OF THIN CYLINDRICAL SHELLS UNDER AXIAL COMPRESSION

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In two previous papers (Ref.1 and Ref.2), the authors have discussed in detail the inadequacy of the classical theory of thin shells in explaining the buckling phenomenon of cylindrical and spherical shells. It was shown that not only the calculated buckling load is 4 to 5 times higher than that experimentally observed, but the buckling wave pattern found is also different from that predicted. Furthermore, it was pointed out that the different explainations for this discrepancy advanced by L.H. Donnell (Ref.3) and W. Flügge (Ref.4) are untenable when certain conclusions drawn from these explainations are compared with the experimental facts. The authors are then led by a theoretical investigation on spherical shells (Ref.1) to the belief that the buckling phenomenon of curved shells can only be explained by means of non-linear large deflection theory. This point of view is further substantiated by a model experiment on thin columns with non-linear elastic support (Ref.2). It is evident from the two cases investigated that the load necessary to keep the shell in equilibrium drops very rapidly with increase in wave amplitude once the shell started to buckle. This

characteristic ~~of dropping~~ load shows, first of all, that there is a release of the elastic energy stored in the shell once the buckling has started, and thus explains the observed rapidity of the buckling process. Furthermore, ~~this characteristic~~ it also brings in the possibility of a decrease in the buckling load when there are slight imperfections in the test specimen and when there are vibrations during the testing process.

In this paper, the authors will show by means of an approximate calculation, that the ~~dropping~~ load ~~characteristic of~~ sustained by the shell drops with increasing deflection in case of a thin uniform cylindrical shell under axial compression. Consequently, they hope that they have thus offered an acceptable explaination of the observed facts.

Stresses in the Median Surface and the Expression for the Total Energy of the System

Let x and y be measured in the axial and the circumferential direction in the median surface of the undeformed cylindrical shell. Let u, v, and w ~~(Fig.)~~ be the components of displacement of a point on the median surface of the shell in the ~~axial~~ x-direction, the ~~circumferential~~ y-direction, and the radial direction. Then the unit strains in the x and y-directions, ε_x, ε_y and the unit shear γ_{xy} at a point in the median surface can be expressed in the following forms, including terms up to second order:

$$
\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\
\varepsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - \frac{w}{R} \qquad (1) \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}
\end{aligned}
$$

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In two previous papers (Ref. 1 and Ref. 2), the authors have discussed in detail the inadequacy of the classical theory of thin shells in explaining the buckling phenomenon of cylindrical and spherical shells. It was shown that not only the calculated buckling load is 3 to 5 times higher than that found by experiments, but the observed wave pattern of the buckled shell is also different from that predicted. Furthermore, it was pointed out that the different explanations for this discrepancy advanced by L. H. Donnell (Ref. 3) and W. Flügge (Ref. 4) are untenable when certain conclusions drawn from these explanations are compared with the experimental facts. The authors are then led by a theoretical investigation on spherical shells (Ref. 1) to the belief that the buckling phenomenon of curved shells can only be explained by means of non-linear large deflection theory. This point of view is further substantiated by a model experiment on thin columns with non-linear elastic support (Ref. 2). It is evident from these two investigations that the load necessary to keep the shell in equilibrium drops very rapidly with increase in wave amplitude once the shell started to buckle. This characteristic shows, first of all, that there is a release of the elastic energy stored in the shell once the buckling has started, and thus explains the observed rapidity of the buckling process. Furthermore, it also brings in the possibility of a decrease in the buckling load when there are slight imperfections in the test specimen and when there are vibrations during the testing process.

In this paper, the authors will show by means of an approximate

~~calculation that in~~ case of a thin uniform cylindrical shell under axial compression, the load sustained by the shell drops with increasing deflection. ~~Consequently, they hope that they have thus offered an acceptable explanation of the observed facts.~~

First it is shown by an approximate calculation that in a uni-

Then the results of this calculation are used for a more detailed discussion of the buckling process as observed in an actual testing machine.

Stresses in the Median Surface and the Expression for the Total Energy of the System

Let x and y be measured in the axial and the circumferential direction in the median surface of the undeformed cylindrical shell and u, v, and w be the components of displacement of a point on the median surface of the shell in the x-direction, the y-direction, and the radial direction. Then the unit strains in the x and y-directions, ε_x, ε_y and the unit shear γ_{xy} at a (arbitrary) point in the median surface can be expressed in the following forms, including terms up to second order:

$$
\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\
\varepsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - \frac{w}{R} \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}
\end{aligned}
\qquad (1)
$$

where R is the radius of the undeformed median surface of the shell. The stresses and the strains in the median surface of the shell are, however, related to each other by the following equations:

$$
\begin{aligned}
\sigma_x &= \frac{E}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_y\right) \\
\sigma_y &= \frac{E}{1-\nu^2}\left(\varepsilon_y + \nu\varepsilon_x\right) \\
\tau_{xy} &= \frac{E}{2(1+\nu)}\gamma_{xy}
\end{aligned}
\qquad (2)
$$

where E is Young's modulus of elasticity and ν is Poisson's ratio. Therefore by substituting Eq. (1) into Eq. (2), the following connections between the components of stress in the median surface and the components of displacement of the median surface are obtained:

THE BUCKLING OF THIN CYLINDRICAL SHELLS UNDER AXIAL COMPRESSION

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In two previous papers (Ref. 1 and Ref. 2), the authors have discussed in detail the inadequacy of the classical theory of thin shells in explaining the buckling phenomenon of cylindrical and spherical shells. It was shown that not only the calculated buckling load is 3 to 5 times higher than that found by experiments, but the observed wave pattern of the buckled shell is also different from that predicted. Furthermore, it was pointed out that the different explanations for this discrepancy advanced by L. H. Donnell (Ref. 3) and W. Flügge (Ref. 4) are untenable when certain conclusions drawn from these explanations are compared with the experimental facts. ~~The authors are then led~~ By a theoretical investigation on spherical shells (Ref. 1) to ~~the belief~~ that the buckling phenomenon of curved shells can only be explained by means of non-linear large deflection theory. This point of view ~~is further~~ substantiated by a model experiment on ~~thin~~ columns with non-linear elastic support (Ref. 2). ~~It is evident from these two investigations that~~ the load necessary to keep the shell in equilibrium drops very rapidly with increase in wave amplitude once the shell started to buckle. This characteristic shows, first of all, that there is a release of the elastic energy stored in the shell once the buckling has started, and thus explains the observed rapidity of the buckling process. Furthermore, it also brings in the possibility of a decrease in the buckling load when there are slight imperfections in the test specimen and when there are vibrations during the testing process.

In this paper, the authors will show by means of an approximate

calculation that in case of a thin uniform cylindrical shell under axial compression, the load sustained by the shell drops with increasing deflection. Consequently, they hope that they have thus offered an acceptable explanation of the observed facts.

Stresses in the Median Surface and the Expression for the Total Energy of the System

Let x and y be measured in the axial and the circumferential direction in the median surface of the undeformed cylindrical shell and u, v, and w be the components of displacement of a point on the median surface of the shell in the x-direction, the y-direction, and the radial direction. Then the unit strains in the x and y-directions, ε_x, ε_y and the unit shear γ_{xy} at a point in the median surface can be expressed in the following forms, including terms up to second order:

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \varepsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - \frac{w}{R} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \end{aligned} \tag{1}$$

where R is the radius of the undeformed median surface of the shell. The stresses and the strains in the median surface of the shell are, however, related to each other by the following equations:

$$\begin{aligned} \sigma_x &= \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) \\ \sigma_y &= \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) \\ \tau_{xy} &= \frac{E}{2(1+\nu)}\gamma_{xy} \end{aligned} \tag{2}$$

where E is Young's modulus of elasticity and ν is Poisson's ratio. Therefore by substituting Eq. (1) into Eq. (2), the following connections between the components of stress in the median surface and the components of displacement of the median surface are obtained:

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In this paper, the same ideas are applied to the case of a thin uniform cylindrical shell under axial compression. First it is shown by an approximate calculation that again the load sustained by the shell drops with increasing

deflection. Then the results of this calculation are used for a more detailed discussion of the buckling process as observed in an actual testing machine.

Stresses in the Median Surface and the Expression for the Total Energy of the System

Let and be measured in the axial and the circumferential direction in the median surface of the undeformed cylindrical shell and , and be the components of displacement of a point on the median surface of the shell in the -direction, the -direction, and the radial direction. Then at an arbitrary point in the median surface the unit strains in the and -directions, , and the unit shear can be expressed in the following forms, including terms up to second order:

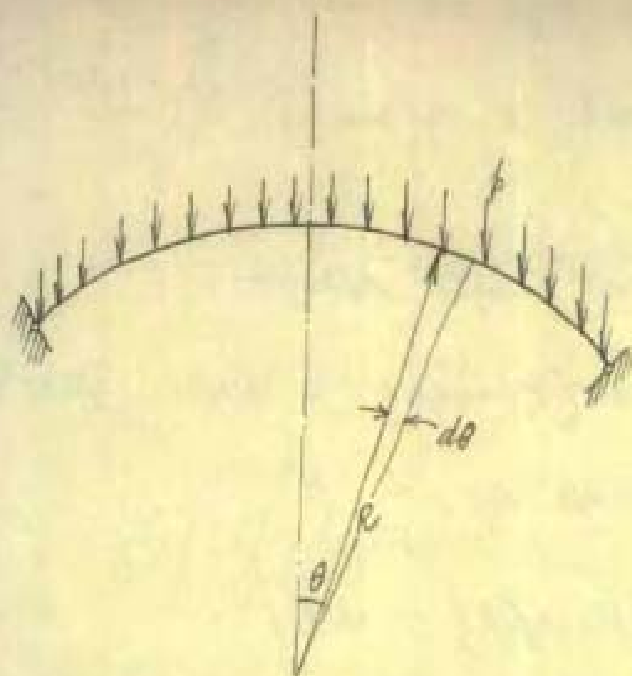
(1)

is the radius of the undeformed median surface of the shell. The stresses and the strains in the median surface of the shell are, however, related to each other by the following equations:

(2)

where is Young's modulus of elasticity and is Poisson's ratio. Therefore, by substituting Eq. (1) into Eq. (2), the following connections between the components of stress in the median surface and the components of displacement of the median surface are obtained:

1)



The original form of the shell is spherical.

Now suppose the deflected form of the shell is axially symmetrical.

$$\theta_1 = \theta_0 + \theta_0 f(\theta_0) = \theta_0 [1 + f(\theta_0)]$$

$$R = R_0 + R\, g(\theta_0)$$

$$= R[1 + g(\theta_0)]$$

The original length of the element $(ds)_0 = R(d\theta)_0$

The new length of the element



$$= \sqrt{R^2(d\theta)^2 + (dR)^2}$$

$$= \sqrt{R^2[1+g(\theta_0)]^2[\{1+f(\theta_0)\}d\theta_0 + \theta_0 f'(\theta_0)\, d\theta_0]^2 + R^2[g'(\theta_0)]^2(d\theta_0)^2}$$

$$= R\sqrt{[1+g(\theta_0)]^2[1+f(\theta_0) + \theta_0 f'(\theta_0)]^2 + [g'(\theta_0)]^2}\; d\theta_0$$

If the deflection is inextensional in the sense that $(ds)_0 = (ds)_1$

then $$\underline{[1+g(\theta_0)]^2[1+f(\theta_0) + \theta_0 f'(\theta_0)]^2 + [g'(\theta_0)]^2 = 1}$$

2)

The distance of the element from the axis is

$$R \sin\theta_0$$

before deflection.

The distance is $R \sin\theta$, after deflection.

$$R[1+g(\theta_0)] \sin[\theta_0(1+f(\theta_0))]$$

$\sin[\theta_0+\theta_0 f(\theta_0)]$

The change in length of the ring ds is

$$2\pi R\left[[1+g(\theta_0)] \sin\{\theta_0(1+f(\theta_0))\} - \sin\theta_0\right]$$

The strain energy stored in this ds is

$$\frac{1}{2}[E\varepsilon]t\, ds\, 2\pi R\left[[1+g(\theta_0)]\sin[\theta_0(1+f(\theta_0))]-\sin\theta_0\right]$$

t = thickness

$\frac{1}{2}E\varepsilon^2 V$

now

$$\varepsilon = [1+g(\theta_0)]\frac{\sin[\theta_0[1+f(\theta_0)]]}{\sin\theta_0} - 1$$

~~$= [1+g(\theta_0)]\{\cos[\theta_0 f(\theta_0)] + \cot\theta_0 \sin$~~

The total strain energy

$$= \frac{1}{2}tER^2 2\pi\int_0^{\alpha}\left\{[1+g(\theta_0)]\frac{\sin\{\theta_0[1+f(\theta_0)]\}}{\sin\theta_0} - 1\right\}^2 \sin\theta_0\, d\theta_0$$

Potential energy of the pressure force.

pV where V = volume under the <u>shell</u>

3

The volume under the shell

$$= \int_0^\alpha \frac{1}{3}\, 2\pi R \sin\theta_1 \cdot r \cdot r\, d\theta_1$$

$$= \frac{2\pi R^3}{3} \int_0^\alpha [1+g(\theta_0)]^3 \sin\{\theta_0[1+f(\theta_0)]\}\{[1+f(\theta_0)]+\theta_0 f'(\theta_0)\}\, d\theta_0$$

The integral to be minimized is

$$\left\{\frac{t}{R}E\right\} \int_0^\alpha \left\{[1+g(\theta_0)]\frac{\sin\{\theta_0[1+f(\theta_0)]\}}{\sin\theta_0} - 1\right\}^2 \sin\theta_0\, d\theta_0$$

$$- \frac{2p}{3} \int_0^\alpha [1+g(\theta_0)]^3 \sin\{\theta_0[1+f(\theta_0)]\} \cdot [1+f(\theta_0)+\theta_0 f'(\theta_0)]\, d\theta_0$$

To simplify the expression, let us put $\theta_0 f(\theta_0) = h(\theta_0)$

$$I = \left\{\frac{t}{R}E\right\} \int_0^\alpha \left\{[1+g(\theta_0)]\frac{\sin(\theta_0+h(\theta_0))}{\sin\theta_0} - 1\right\}^2 \sin\theta_0\, d\theta_0$$

$$- \frac{2p}{3} \int_0^\alpha [1+g(\theta_0)]^3 \sin(\theta_0+h(\theta_0)) \cdot [1+h'(\theta_0)]\, d\theta_0$$

The inextensional condition is

$$[1+g(\theta_0)]^2[1+h'(\theta_0)]^2 + [g'(\theta_0)]^2 - 1 = 0$$

$$\varepsilon_1 = u_0 = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \qquad \text{—(1)}$$

$$\varepsilon_2 = 0 = \frac{1}{a}\frac{\partial v}{\partial \theta} - \frac{w}{a} + \frac{1}{2a^2}\left(\frac{\partial w}{\partial \theta}\right)^2 \qquad \text{—(2)}$$

$$\gamma = 0 = \frac{1}{a}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} + \frac{1}{a}\frac{\partial w}{\partial x}\frac{\partial w}{\partial \theta} \qquad \text{—(3)}$$

Differentiate (1) with respect to $\frac{1}{a}\frac{\partial}{\partial \theta}$ & (3) $\frac{\partial}{\partial x}$,

$$0 = \frac{1}{a}\frac{\partial^2 u}{\partial x \partial \theta} + \frac{1}{a}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial \theta}$$

$$0 = \frac{1}{a}\frac{\partial^2 u}{\partial x \partial \theta} + \frac{\partial^2 v}{\partial x^2} + \frac{1}{a}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial \theta} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial \theta}\right)$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{1}{a}\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial \theta} = 0 \qquad \text{—(4)}$$

Differentiate (4) with respect to $\frac{1}{a}\frac{\partial}{\partial \theta}$ and (2) $\frac{\partial^2}{\partial x^2}$,

$$\frac{1}{a}\frac{\partial^3 v}{\partial x^2 \partial \theta} + \frac{1}{a^2}\left[\frac{\partial^3 w}{\partial x^2 \partial \theta}\frac{\partial w}{\partial \theta} + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial \theta^2}\right] = 0$$

$$\frac{1}{a}\frac{\partial^3 v}{\partial x^2 \partial \theta} - \frac{1}{a}\frac{\partial^2 w}{\partial x^2} + \frac{1}{a^2}\frac{\partial}{\partial x}\left[\frac{\partial w}{\partial \theta}\frac{\partial^2 w}{\partial x \partial \theta}\right]$$

$$= \frac{1}{a}\frac{\partial^3 v}{\partial x^2 \partial \theta} - \frac{1}{a}\frac{\partial^2 w}{\partial x^2} + \frac{1}{a^2}\left[\left(\frac{\partial^2 w}{\partial x \partial \theta}\right)^2 + \frac{\partial w}{\partial \theta}\frac{\partial^3 w}{\partial x^2 \partial \theta}\right]$$

$$\boxed{\frac{1}{a^2}\left(\frac{\partial^2 w}{\partial x \partial \theta}\right)^2 - \frac{1}{a^2}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial \theta^2} - \frac{1}{a}\frac{\partial^2 w}{\partial x^2} = 0}$$

Or

$$\boxed{(s^2 - rt) - ar = 0}$$

118)

We see that the w- deflection for region 1

$$w = kr\left[a^2-\left(\theta^2+\frac{x^2}{r^2}\right)\right]$$

satisfies the partial differential equation

$$-\frac{4k^2}{r^2}+\frac{2k}{r^2}=0$$

or $\quad k=\frac{1}{2}$

$$\frac{\partial w}{\partial x}=-\left(\frac{x}{r}\right)$$

Thus $\quad w=\frac{r}{2}\left[a^2-\left(\theta^2+\frac{x^2}{r^2}\right)\right]$, $\quad \frac{1}{r}\frac{\partial w}{\partial \theta}=-\theta$

We have the relation $\quad \frac{\partial u}{\partial x}=-\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$

$$=-\frac{1}{2}\left(\frac{x}{r}\right)^2$$

$$\frac{u}{r}=f(\theta)-\frac{1}{6}\left(\frac{x}{r}\right)^3$$

Also

$$\frac{1}{r}\frac{\partial v}{\partial \theta}=\frac{w}{r}-\frac{1}{2}\left(\frac{\partial w}{\partial \theta}\right)^2\frac{1}{r^2}=\frac{1}{2}\left[a^2-\left(\theta^2+\frac{x^2}{r^2}\right)\right]-\frac{1}{2}\theta^2$$

$$=\frac{1}{2}a^2-\theta^2-\frac{1}{2}\frac{x^2}{r^2}$$

$$\frac{v}{r}=\frac{1}{2}a^2\theta-\frac{1}{3}\theta^3-\frac{1}{2}\frac{x^2}{r^2}\theta+g\left(\frac{x}{r}\right)$$

Now $\quad -\frac{1}{r}\frac{\partial w}{\partial x}\frac{\partial w}{\partial \theta}=\frac{1}{r}\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial x}$ or

$$-\theta\left(\frac{x}{r}\right)=f'(\theta)-\left(\frac{x}{r}\right)\theta+g'\left(\frac{x}{r}\right)$$

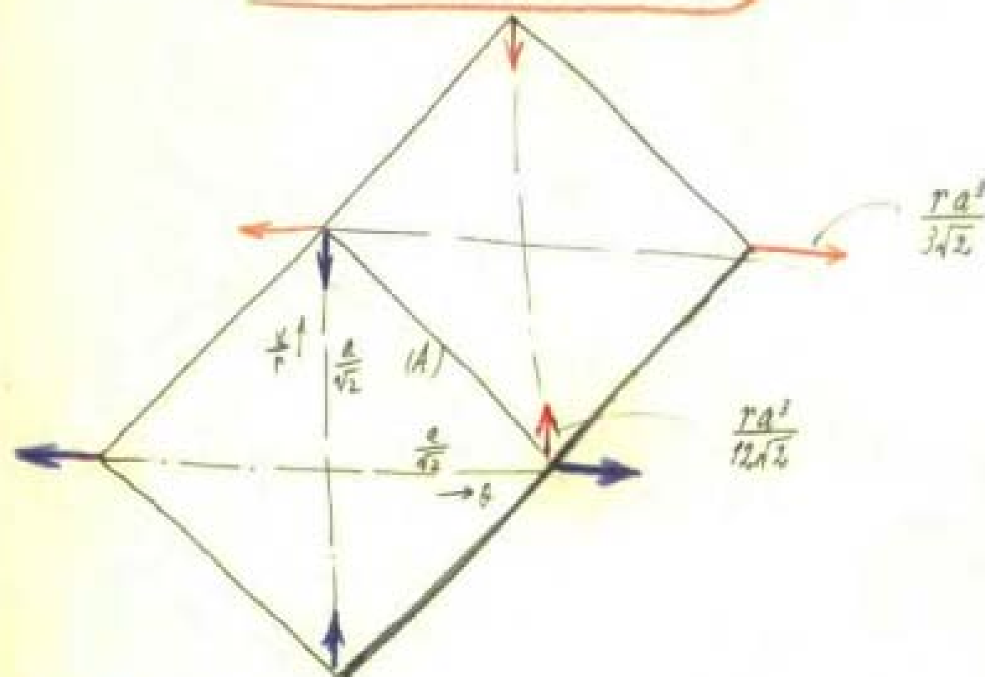
Thus $\quad f'(\theta)=0=g'\left(\frac{x}{r}\right)$, $\therefore$ put $f(\theta)=0=g\left(\frac{x}{r}\right)$

Thus

$$w = \frac{p}{2}\left[a^2 - \left(\theta^2 + \frac{x^2}{r^2}\right)\right]$$

$$u = -\frac{p}{6}\left(\frac{x}{r}\right)^3$$

$$v = \frac{p}{2}\left(a^2 - \frac{x^2}{r^2}\right)\theta - \frac{p}{3}\theta^3$$



Along the boundary (A), $\frac{a}{\sqrt{2}} = \theta + \left(\frac{x}{r}\right)$

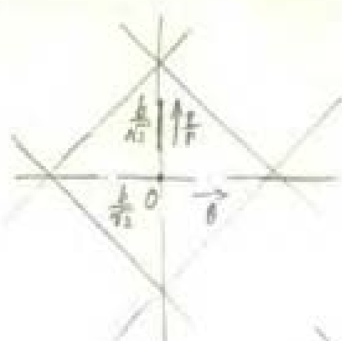
$$v = \frac{p}{2}\left(a^2 - \frac{x^2}{r^2}\right)\left(\frac{a}{\sqrt{2}} - \frac{x}{r}\right) - \frac{p}{3}\left(\frac{a}{\sqrt{2}} - \frac{x}{r}\right)^3$$

$$\frac{v}{p} = \frac{1}{2}\left\{\frac{a^3}{\sqrt{2}} - a^2\left(\frac{x}{r}\right) - \frac{a}{\sqrt{2}}\left(\frac{x}{r}\right)^2 + \left(\frac{x}{r}\right)^3\right\}$$

$$- \frac{1}{3}\left\{\frac{a^3}{2\sqrt{2}} - \frac{3}{2}a^2\left(\frac{x}{r}\right) + \frac{3}{\sqrt{2}}a\left(\frac{x}{r}\right)^2 - \left(\frac{x}{r}\right)^3\right\}$$

$$= \frac{a^3}{3\sqrt{2}} - \frac{3}{2\sqrt{2}}a\left(\frac{x}{r}\right)^2 + \frac{5}{6}\left(\frac{x}{r}\right)^3$$

The u, v in region (IV) can be expressed as 120)



$$u = \frac{p}{6}\left(\frac{a}{b}\right)^3\left(\frac{x}{r}\right)^3 - pk\left(\frac{x}{r}\right)$$

$$v = \left(\frac{a}{b}\right)^3\left\{\frac{p}{2}\left(b^2 - \frac{x^2}{r^2}\right)\theta - \frac{p}{3}\theta^3\right\}$$

The u, v in the rectangular region can be obtained by returning the ◇ figure 90° counter-clockwise + than make v negative.

$$u = -\left\{\frac{\gamma}{6}\theta^3 - \frac{\gamma k}{2}\theta\right\}$$

$$v = -\left\{\frac{\gamma}{2}(a^2-\theta^2)\left(\frac{y}{r}\right) - \frac{\gamma}{3}\left(\frac{y}{r}\right)^3\right\}$$

We then shrink one side by the ratio $\left(\frac{b}{a}\right)$

$$\theta = \left(\frac{a}{b}\right)\frac{s}{\sqrt{2}} - \frac{t}{\sqrt{2}}$$

$$\frac{y}{r} = \left(\frac{a}{b}\right)\frac{s}{\sqrt{2}} + \frac{t}{\sqrt{2}}$$

$$\bar{u} = -\frac{u}{\sqrt{2}} - \frac{v}{\sqrt{2}}, \qquad \bar{v} = \frac{u}{\sqrt{2}} + \frac{v}{\sqrt{2}}$$

Cont'd to p. 677 !!!

$$\frac{w}{R} = \left(f_0+\frac{1}{4}f_1\right) + \frac{1}{2}f_1\left[\cos\frac{mx}{R}\cos\frac{ny}{R} + \frac{1}{4}\cos\frac{2mx}{R} + \frac{1}{4}\cos\frac{2ny}{R}\right]$$

$$+ \frac{1}{4}f_2\left[\cos\frac{2mx}{R} + \cos\frac{2ny}{R}\right]$$

$$\frac{w}{R} = \left(f_0+\frac{1}{4}f_1\right) + \frac{1}{2}f_1\cos\frac{mx}{R}\cos\frac{ny}{R} + \frac{1}{4}\left(\frac{1}{2}f_1+f_2\right)\cos\frac{2mx}{R} + \frac{1}{4}\left(\frac{1}{2}f_1+f_2\right)\cos\frac{2ny}{R}$$

$$\frac{\partial w}{\partial x} = -n\left[\frac{1}{2}\left(\frac{m}{n}\right)f_1\sin\frac{mx}{R}\cos\frac{ny}{R} + \frac{1}{2}\left(\frac{2m}{n}\right)\left(\frac{1}{2}f_1+f_2\right)\sin\frac{2mx}{R}\right]$$

$$\frac{\partial w}{\partial y} = -n\left[\frac{1}{2}f_1\cos\frac{mx}{R}\sin\frac{ny}{R} + \frac{1}{2}\left(\frac{1}{2}f_1+f_2\right)\sin\frac{2ny}{R}\right]$$

$$\frac{1}{R}\frac{\partial^2 w}{\partial x^2} = -\left(\frac{n}{R}\right)^2\left[\frac{1}{2}\left(\frac{m}{n}\right)^2 f_1\cos\frac{mx}{R}\cos\frac{ny}{R} + \left(\frac{m}{n}\right)^2\left(\frac{1}{2}f_1+f_2\right)\cos\frac{2mx}{R}\right]$$

$$\frac{1}{R}\frac{\partial^2 w}{\partial y^2} = -\left(\frac{n}{R}\right)^2\left[\frac{1}{2}f_1\cos\frac{mx}{R}\cos\frac{ny}{R} + \left(\frac{1}{2}f_1+f_2\right)\cos\frac{2ny}{R}\right]$$

$$\frac{1}{R}\frac{\partial^2 w}{\partial x\partial y} = +\left(\frac{n}{R}\right)^2\left[\frac{1}{2}\left(\frac{m}{n}\right)\sin\frac{mx}{R}\sin\frac{ny}{R}\right] \qquad \left(\mu = \frac{m}{n}\right)$$

$$\Delta\Delta F = E\left(\frac{n}{R}\right)^2\Bigg[n^2\Bigg\{-\frac{1}{8}\mu^2 f_1^2\left(\cos\frac{2mx}{R}+\cos\frac{2ny}{R}\right) - \frac{1}{4}\mu^2 f_1\left(\frac{1}{2}f_1+f_2\right)\left(\cos\frac{mx}{R}+\cos\frac{3mx}{R}\right)\cos\frac{ny}{R}$$

$$-\frac{1}{4}\mu^2 f_1\left(\frac{1}{2}f_1+f_2\right)\cos\frac{mx}{R}\left(\cos\frac{ny}{R}+\cos\frac{3ny}{R}\right)$$

$$-\mu^2\left(\frac{1}{2}f_1+f_2\right)^2\cos\frac{2mx}{R}\cos\frac{2ny}{R} + \frac{1}{2}\mu^2 f_1\cos\frac{mx}{R}\cos\frac{ny}{R} + \mu^2\left(\frac{1}{2}f_1+f_2\right)\cos\frac{2mx}{R}\Bigg]$$

$$= -E\mu^2\left(\frac{n}{R}\right)^2\Bigg[\left\{\frac{1}{8}f_1^2 n^2 - \left(\frac{1}{2}f_1+f_2\right)\right\}\cos\frac{2mx}{R} + \frac{1}{4}f_1\left(\frac{1}{2}f_1+f_2\right)n^2\cos\frac{3mx}{R}\cos\frac{ny}{R}$$

$$+\frac{1}{4}f_1\left(\frac{1}{2}f_1+f_2\right)n^2\cos\frac{mx}{R}\cos\frac{3ny}{R} + \left\{\frac{1}{2}f_1\left(\frac{1}{2}f_1+f_2\right)n^2 - \frac{1}{2}f_1\right\}\cos\frac{mx}{R}\cos\frac{ny}{R}$$

$$+\left(\frac{1}{2}f_1+f_2\right)^2 n^2\cos\frac{2mx}{R}\cos\frac{2ny}{R} + \frac{1}{8}f_1^2 n^2\cos\frac{2ny}{R}\Bigg]$$

$$F = -E\mu^2\left(\frac{R}{n}\right)^2\Big[\frac{1}{(3\mu)^4}\left\{\frac{1}{8}f_1^2n^2-\left(\frac{1}{2}f_1+f_2\right)\right\}\cos\frac{2mx}{R}+\frac{1}{2^4}\frac{1}{8}f_1^2n^2\cos\frac{2ny}{R}$$

$$+\frac{1}{(\mu^2+1)^2}\left\{\frac{1}{2}f_1\left(\frac{1}{2}f_1+f_2\right)n^2-\frac{1}{2}f_1\right\}\cos\frac{mx}{R}\cos\frac{ny}{R}+\frac{1}{4}\frac{1}{(9\mu^2+1)^2}f_1\left(\frac{1}{2}f_1+f_2\right)n^2\cos\frac{3mx}{R}\cos\frac{ny}{R}$$

$$+\frac{1}{4}\frac{1}{(\mu^2+9)^2}f_1\left(\frac{1}{2}f_1+f_2\right)n^2\cos\frac{mx}{R}\cos\frac{3ny}{R}+\frac{1}{16(\mu^2+1)^2}\left(\frac{1}{2}f_1+f_2\right)^2n^2\cos\frac{2mx}{R}\cos\frac{2ny}{R}\Big]$$

$$\sigma_x = E\mu^2\Big[\frac{1}{32}f_1^2n^2\cos\frac{2ny}{R}+\frac{1}{(\mu^2+1)^2}\left\{\frac{1}{2}f_1\left(\frac{1}{2}f_1+f_2\right)n^2-\frac{1}{2}f_1\right\}\cos\frac{mx}{R}\cos\frac{ny}{R}$$

$$+\frac{1}{4}\frac{1}{(9\mu^2+1)^2}f_1\left(\frac{1}{2}f_1+f_2\right)n^2\cos\frac{3mx}{R}\cos\frac{ny}{R}+\frac{1}{4}\frac{9}{(\mu^2+9)^2}f_1\left(\frac{1}{2}f_1+f_2\right)n^2\cos\frac{mx}{R}\cos\frac{3ny}{R}$$

$$+\frac{1}{4}\frac{1}{(\mu^2+1)^2}\left(\frac{1}{2}f_1+f_2\right)^2n^2\cos\frac{2mx}{R}\cos\frac{2ny}{R}\Big]$$

$$\sigma_y = E\mu^2\Big[\frac{1}{(3\mu)^2}\left\{\frac{1}{8}f_1^2n^2-\left(\frac{1}{2}f_1+f_2\right)\right\}\cos\frac{2mx}{R}+\frac{n^2}{(\mu^2+1)^2}\left\{\frac{1}{2}f_1\left(\frac{1}{2}f_1+f_2\right)n^2-\frac{1}{2}f_1\right\}\cos\frac{mx}{R}\cos\frac{ny}{R}$$

$$+\frac{1}{4}\frac{9\mu^2}{(9\mu^2+1)^2}f_1\left(\frac{1}{2}f_1+f_2\right)n^2\cos\frac{3mx}{R}\cos\frac{ny}{R}+\frac{1}{4}\frac{\mu^2}{(\mu^2+9)^2}f_1\left(\frac{1}{2}f_1+f_2\right)n^2\cos\frac{mx}{R}\cos\frac{3ny}{R}$$

$$+\frac{1}{4}\frac{\mu^2}{(\mu^2+1)^2}\left(\frac{1}{2}f_1+f_2\right)^2n^2\cos\frac{2mx}{R}\cos\frac{2ny}{R}\Big]$$

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$$R_1=\mu^4\Big[\frac{2}{(3\mu)^4}\Big\{\frac{1}{8}f_1^2n^2-\Big(\frac{1}{2}f_1+f_2\Big)\Big\}^2+\frac{1}{512}f_1^4n^4+\frac{1}{4(\mu^2+1)^2}\Big\{f_1\Big(\frac{1}{2}f_1+f_2\Big)n^2-f_1\Big\}^2$$

$$+\frac{1}{16}\frac{1}{(9\mu^2+1)^2}f_1^2n^4\Big(\frac{1}{2}f_1+f_2\Big)^2+\frac{1}{16}\frac{1}{(\mu^2+9)^2}f_1^2n^4\Big(\frac{1}{2}f_1+f_2\Big)^2+\frac{1}{16}\frac{1}{(\mu^2+1)^2}\Big(\frac{1}{2}f_1+f_2\Big)^2n^4\Big]$$

$$=\frac{\mu^4}{4}\Big[\frac{1}{2\mu^4}\Big\{\frac{1}{64}f_1^4n^4-\Big(\frac{1}{8}f_1^3n^2+\frac{1}{4}f_1^2f_2n^2\Big)+\Big(\frac{1}{4}f_1^2+f_1f_2+f_2^2\Big)\Big\}+\frac{1}{128}f_1^4n^4$$

$$+\frac{1}{(\mu^2+1)^2}\Big\{\Big(\frac{1}{4}f_1^4n^4+f_1^3f_2n^4+f_1^2f_2^2n^4\Big)-\Big(f_1^3n^2+2f_1^2f_2n^2\Big)+f_1^2\Big\}$$

$$+\frac{1}{4}\Big\{\frac{1}{(9\mu^2+1)^2}+\frac{1}{(\mu^2+9)^2}\Big\}\Big\{\frac{1}{4}f_1^4n^4+f_1^3f_2n^4+f_1^2f_2^2n^4\Big\}+\frac{1}{4}\frac{1}{(\mu^2+1)^2}\Big\{\frac{1}{16}f_1^4n^4+\frac{1}{2}f_1^3f_2n^4+\frac{3}{2}f_1^2f_2^2n^4$$

$$+2f_1f_2^3n^4+f_2^4n^4\Big\}\Big]$$

$$=\frac{\mu^4}{4}\Big[n^4\Big\{f_1^4\Big(\frac{1}{128\mu^4}+\frac{1}{128}+\frac{1}{4(\mu^2+1)^2}+\frac{1}{16(9\mu^2+1)^2}+\frac{1}{16(\mu^2+9)^2}+\frac{1}{64(\mu^2+1)^2}\Big)$$

$$+f_1^3f_2\Big(\frac{1}{(\mu^2+1)^2}+\frac{1}{4(9\mu^2+1)^2}+\frac{1}{4(\mu^2+9)^2}+\frac{1}{8(\mu^2+1)^2}\Big)$$

$$+f_1^2f_2^2\Big(\frac{1}{(\mu^2+1)^2}+\frac{1}{4(9\mu^2+1)^2}+\frac{1}{4(\mu^2+9)^2}+\frac{3}{8(\mu^2+1)^2}\Big)+\frac{1}{2(\mu^2+1)^2}f_1f_2^3+\frac{1}{4(\mu^2+1)^2}f_2^4\Big\}$$

$$-n^2\left\{f_1^3\left(\frac{1}{16\mu^4}+\frac{1}{(\mu^2+1)^2}\right)+f_1^2f_2\left(\frac{1}{8\mu^4}+\frac{2}{(\mu^2+1)^2}\right)\right\}+\left\{f_1^2\left(\frac{1}{8\mu^4}+\frac{1}{(\mu^2+1)^2}\right)\right.$$

$$\left.+\frac{1}{2\mu^4}f_1f_2+\frac{1}{2\mu^4}f_2^2\right\}$$

$$P_1=\frac{1}{4}\left\{n^4\left[\left\{\frac{(1+\mu^4)}{128}+\frac{17}{64}\frac{\mu^4}{(\mu^2+1)^2}+\frac{\mu^4}{16(9\mu^2+1)^2}+\frac{\mu^4}{16(\mu^2+9)^2}\right\}f_1^4\right.\right.$$

$$+\left\{\frac{9}{8}\frac{\mu^4}{(\mu^2+1)^2}+\frac{\mu^4}{4(9\mu^2+1)^2}+\frac{\mu^4}{4(\mu^2+9)^2}\right\}f_1^3f_2+\left\{\frac{11}{8}\frac{\mu^4}{(\mu^2+1)^2}+\frac{\mu^4}{4(9\mu^2+1)^2}+\frac{\mu^4}{4(\mu^2+9)^2}\right\}f_1^2f_2^2$$

$$\left.+\frac{\mu^4}{2(\mu^2+1)^2}f_1f_2^3+\frac{\mu^4}{4(\mu^2+1)^2}f_2^4\right]$$

$$\left.-n^2\left[\left\{\frac{1}{16}+\frac{\mu^4}{(\mu^2+1)^2}\right\}f_1^3+\left\{\frac{1}{8}+\frac{2\mu^4}{(\mu^2+1)^2}\right\}f_1^2f_2\right]+\left[\left\{\frac{1}{8}+\frac{\mu^4}{(\mu^2+1)^2}\right\}f_1^2+\frac{1}{2}f_1f_2+\frac{1}{2}f_2^2\right]\right\}$$

$$P_2=\frac{1}{12(1-\nu^2)}\left(\frac{h}{R}\right)^2n^4\left\{\frac{1}{4}(1+\mu^2)^2f_1^2+2\mu^4\left(\frac{1}{2}f_1+f_2\right)^2+2\left(\frac{1}{2}f_1+f_2\right)^2\right\}$$

$$P_2=\frac{1}{6(1-\nu^2)}\left(\frac{h}{R}\right)^2n^4\left\{f_1^2\left[\frac{1}{8}(1+\mu^2)^2+\frac{1}{4}(1+\mu^4)\right]+(1+\mu^4)f_1f_2+(1+\mu^4)f_2^2\right\}$$

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$$A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 = 0$$

$$A_3 = (\gamma\xi)^2\left\{\frac{3\mu^4}{(\mu^2+1)^2} + \frac{\mu^4}{(9\mu^2+1)^2} + \frac{\mu^4}{(\mu^2+9)^2}\right\}$$

$$A_2 = (\gamma\xi)^2\left\{\frac{9}{2}\frac{\mu^4}{(\mu^2+1)^2} + \frac{3}{2}\frac{\mu^4}{(9\mu^2+1)^2} + \frac{3}{2}\frac{\mu^4}{(\mu^2+9)^2}\right\} - (\gamma\xi)\left\{\frac{1}{2} + \frac{8\mu^4}{(\mu^2+1)^2}\right\}$$

$$A_1 = (\gamma\xi)^2\left\{\frac{1+\mu^4}{16} + \frac{1}{4}\frac{\mu^4}{(\mu^2+1)^2} + \frac{1}{4}\frac{\mu^4}{(9\mu^2+1)^2} + \frac{1}{4}\frac{\mu^4}{(\mu^2+9)^2}\right\} - (\gamma\xi)\left\{\frac{1}{2} + \frac{8\mu^4}{(\mu^2+1)^2}\right\} + \left\{\frac{4\mu^4}{(\mu^2+1)^2} - 1\right\}$$

$$- \frac{2}{3(1-\nu^2)}\gamma^2\left\{2(1+\mu^4) - \frac{1}{2}(1+\mu^2)^2\right\}$$

$$A_0 = (\gamma\xi)^2\left\{\frac{1+\mu^4}{32} - \frac{5}{8}\frac{\mu^4}{(\mu^2+1)^2} - \frac{1}{8}\frac{\mu^4}{(9\mu^2+1)^2} - \frac{1}{8}\frac{\mu^4}{(\mu^2+9)^2}\right\} + \left\{\frac{2\mu^4}{(\mu^2+1)^2} - \frac{1}{2}\right\}$$

$$- \frac{2}{3(1-\nu^2)}\gamma^2\left\{(1+\mu^4) - \frac{3}{4}(1+\mu^2)^2\right\}$$

$\frac{7}{40}$

(2)

$$-2\nu\frac{\sigma}{E}-2\left(f_0+\frac{1}{4}f_1\right)+2n^2\left\{\frac{3}{64}f_1^2+\frac{1}{16}f_1f_2+\frac{1}{16}f_2^2\right\}=0$$

$$K=-4\Big[(1-\nu^2)\left(\frac{\sigma}{E}\right)^2+n^2\left\{\frac{3}{32}(\mu^2+\nu)f_1^2+\frac{1}{8}(\mu^2+\nu)f_1f_2+\frac{1}{8}(\mu^2+\nu)f_2^2\right\}\frac{\sigma}{E}$$

$$-n^4\left\{\frac{3}{64}f_1^2+\frac{1}{16}f_1f_2+\frac{1}{16}f_2^2\right\}^2+\left(f_0+\frac{1}{4}f_1\right)^2\Big]$$

$$K=-4\left(\frac{\sigma}{E}\right)^2-n^2\mu^2\left\{\frac{3}{8}f_1^2+\frac{1}{2}f_1f_2+\frac{1}{2}f_2^2\right\}$$

$$\varepsilon=(1-\nu^2)\frac{\sigma}{E}-\nu\left(f_0+\frac{1}{4}f_1\right)+n^2(\mu^2+\nu)\left\{\frac{3}{64}f_1^2+\frac{1}{16}f_1f_2+\frac{1}{16}f_2^2\right\}$$

$$-\nu^2\frac{\sigma}{E}-\nu\left(f_0+\frac{1}{4}f_1\right)+n^2\nu\left\{\frac{3}{64}f_1^2+\frac{1}{16}f_1f_2+\frac{1}{16}f_2^2\right\}=0$$

$$\varepsilon=\frac{\sigma}{E}+n^2\mu^2\left\{\frac{3}{64}f_1^2+\frac{1}{16}f_1f_2+\frac{1}{16}f_2^2\right\}$$

$$\boxed{\frac{\varepsilon R}{t}=\frac{\sigma R}{Et}+\frac{1}{16}\mu^2(f\xi)\xi\left\{\lambda^2+\lambda+0.75\right\}}$$

$$\boxed{\mu = 0.5, \quad \gamma = 0.289, \quad \xi = 3}$$

$$\gamma^2 = 0.083521, \qquad (\gamma\xi) = 0.867, \qquad (\gamma\xi)^2 = 0.751689$$

$$0.0951996228\lambda^3 - 0.56814457\lambda^2 - 1.57447729\lambda - 0.45556880 = 0$$

$$F(-\lambda) = \lambda^3 + 5.9678887\lambda^2 - 16.538697\lambda + 4.785405 = 0$$

$$F'(-\lambda) = 3\lambda^2 + 11.935774\lambda - 16.537697$$

$$F(0.33\ \ ) = +0.013475 \qquad F'(0.33) = 12.273$$

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$$F(0.331098) = +0.000007$$

$$\lambda = -0.331099$$

$$\frac{\sigma R}{Et} = 0.7190407 + \frac{4}{0.289}\left\{0.03131680\lambda^2 - 0.14641820\lambda - 0.08851907\right\}$$

$$= 0.4886323$$

$$\frac{\varepsilon R}{t} = \underline{\underline{0.5101}}$$

$$\xi = 0.4886323^2 + 9\left[0.0046985(-\lambda)^4 - 0.0009396(-\lambda)^3 + 0.0379763(-\lambda)^2 - 0.0363970(-\lambda) + 0.0074196\right]$$

$$= 0.3642 \qquad \Phi = -0.117151$$

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$\cos\frac{\pi x}{R}\cos\frac{\pi y}{R}$

$\frac{\pi x}{R}$	$\frac{\pi y}{R}=0$ $\cos\frac{\pi x}{R}$	$\frac{\pi y}{R}=15^\circ$ $\frac{w}{R}$	$\frac{\pi y}{R}=30^\circ$ $\frac{w}{R}$	$\frac{\pi y}{R}=45^\circ$ $\frac{w}{R}$	$\frac{\pi y}{R}=60^\circ$ $\frac{w}{R}$	$\frac{\pi y}{R}=75^\circ$ $\frac{w}{R}$	
0	1.0000	0.9659	0.8660	0.7071	0.50000	0.2588	
15°	0.9659	0.9330	0.8365	0.6829	0.4830	0.2500	
30°	0.8660	0.8365	0.7500	0.6123	0.4330	0.2241	
45°	0.7071	0.6830	0.6123	0.50000	0.3536	0.1830	
60°	0.5000	0.4830	0.4330	0.3536	0.2500	0.1294	
75°	0.2588	0.2500	0.2241	0.1830	0.1294	0.0670	
90°	0	0	0	0	0	0	

$$\frac{w_1}{R}=\cos^2\frac{\pi(x+y)}{2R}\cos^2\frac{\pi(x-y)}{2R}$$

$\frac{\pi x}{R}$	$\frac{\pi y}{R}=0$ $\frac{w_1}{R}$	$\cos^2\frac{\pi x}{2R}$	$\frac{\pi y}{R}=15^\circ$ $\frac{w_1}{R}$	$\frac{\pi y}{R}=30^\circ$ $\frac{w_1}{R}$	$\frac{\pi y}{R}=45^\circ$ $\frac{w_1}{R}$	$\frac{\pi y}{R}=60^\circ$ $\frac{w_1}{R}$	$\frac{\pi y}{R}=75^\circ$ $\frac{w_1}{R}$	$\frac{\pi y}{R}=90^\circ$ $\frac{w_1}{R}$
0	1.0000	1.0000	0.9660	0.8704	0.7286	0.5625	0.3962	0.25000
15°	0.9660	0.9829	0.9330	0.8390	0.6998	0.5373	0.3750	0.2333
30°	0.8704	0.9330	0.8390	0.7500	0.6187	0.4615	0.3163	0.1875
45°	0.7286	0.8536	0.6998	0.6187	0.50000	0.3643	0.2333	0.1250
60°	0.5625	0.7500	0.5373	0.4665	0.3643	0.2500	0.1440	0.0625
75°	0.3962	0.6295	0.3750	0.3163	0.2333	0.1440	0.0670	0.0167
90°	0.2500	0.50000	0.2333	0.1875	0.1250	0.0625	0.0167	0
105°	0.1373	0.3706	0.1250	0.0922	0.0503	0.0145	0	0.0167
120°	0.0625	0.2500	0.0543	0.0335	0.0107	0	0.0145	0.0625
135°	0.0215	0.1465	0.0168	0.0063	0	0.0107	0.0503	0.1250
150°	0.0045	0.0670	0.0025	0	0.0063	0.0335	0.0922	0.1875
165°	0.0003	0.0170	0	0.0025	0.0168	0.0543	0.1250	0.2333
180°	0	0	0.0003	0.0045	0.0215	0.0625	0.1373	0.2500

$$\frac{w_3}{R} = \frac{1}{4}\left[\cos\frac{2\pi x}{R} + \cos\frac{2\pi y}{R}\right],$$!!! Not Used !!!

$\frac{\pi x}{R}$	$\frac{\pi y}{R}=0$ $\frac{w_3}{R}$	$\frac{\pi y}{R}=15°$ $\frac{w_3}{R}$	$\frac{\pi y}{R}=30°$ $\frac{w_3}{R}$	$\frac{\pi y}{R}=45°$ $\frac{w_3}{R}$	$\frac{\pi y}{R}=60°$ $\frac{w_3}{R}$	$\frac{\pi y}{R}=75°$ $\frac{w_3}{R}$	$\frac{\pi y}{R}=90°$ $\frac{w_3}{R}$
0	0.50000	0.4665	0.37500	0.25000	0.1250	0.0335	0
15°	0.4665	0.4330	0.3415	0.2165	0.0915	0	−0.0335
30°	0.3750	0.3415	0.2500	0.1250	0	−0.0915	−0.1250
45°	0.2500	0.2165	0.1250	0	−0.1250	−0.2165	−0.2500
60°	0.1250	0.0915	0	−0.1250	−0.2500	−0.3415	−0.3750
75°	0.0335	0	−0.0915	−0.2165	−0.3415	−0.4330	−0.4665
90°	0	−0.0335	−0.1250	−0.2500	−0.3750	−0.4665	−0.5000
105°	0.0335	0	−0.0915	−0.2165	−0.3415	−0.4330	−0.4665
120°	0.1250	0.0915	0	−0.1250	−0.2500	−0.3415	−0.3750
135°	0.2500	0.2165	0.1250	0	−0.1250	−0.2165	−0.2500
150°	0.3750	0.3415	0.2500	0.1250	0	−0.0915	−0.1250
165°	0.4665	0.4330	0.3415	0.2165	0.0915	0	−0.0335
180°	0.5000	0.4660	0.3750	0.2500	0.1250	0.0335	0

$$\frac{w}{R} = \left(f_0 + \frac{1}{4}f_1\right) + \left(\frac{1}{2}f_1 + f_2\right)\left[\cos\frac{\pi x}{R}\cos\frac{\pi y}{R} + \frac{1}{4}\cos\frac{2\pi x}{R} + \frac{1}{4}\cos\frac{2\pi y}{R}\right]$$

$$- f_2\cos\frac{\pi x}{R}\cos\frac{\pi y}{R}$$

$$= \left(f_0 - \frac{1}{2}f_2\right) + \left(f_1 + 2f_2\right)\cos^2\frac{(\pi x+\pi y)}{2R}\cos^2\frac{(\pi x-\pi y)}{2R}$$

$$- f_2\cos\frac{\pi x}{R}\cos\frac{\pi y}{R}$$

Amplitude ratio $\quad \frac{w_1}{w_2} = \frac{f_1+2f_2}{-f_2} = -\frac{1+2\rho}{\rho}$

$w_1 : w_2 = 1 : -\rho/1+2\rho$

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$\mu = 1.000$; $n = 26$, $\xi = 0.5$

$\rho = -0.2667875$; $w_1 : w_2 = 1 : 0.57198$

$\frac{nx}{R}$	ny/R						
	0	15°	30°	45°	60°	75°	90°
0	1.5720	1.5185	1.3657	1.1330	0.8485	0.5442	0.25000
15°	1.5185	1.4667	1.3175	1.0905	0.8136	0.5180	0.2333
30°	1.3657	1.3175	1.1790	0.9689	0.7142	0.4447	0.1875
45°	1.1330	1.0905	0.9689	0.7860	0.5666	0.3360	0.1250
60°	0.8485	0.8136	0.7142	0.5666	0.3930	0.2180	0.0625
75°	0.5442	0.5180	0.4447	0.3360	0.2180	0.1053	0.0167
90°	0.2500	0.2330	0.1875	0.1250	0.0625	0.0167	0
105°	−0.0107	−0.0180	−0.0360	−0.0544	−0.0595	−0.0383	0.0167
120°	−0.2235	−0.2220	−0.2142	−0.1916	−0.1430	−0.0595	0.0625
135°	−0.3829	−0.3739	−0.3437	−0.2860	−0.1916	−0.0544	0.1250
150°	−0.4908	−0.4760	−0.4290	−0.3439	−0.2229	−0.0360	0.1875
165°	−0.5522	−0.5337	−0.4760	−0.3738	−0.2397	−0.0180	0.2333
180°	−0.5720	−0.5522	−0.4908	−0.3829	−0.2235	−0.0107	0.2500

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喷 气 推 进

2.1 有关火箭研究的文献调研和分析计算

作者在1937年春天参加了由他的同学F. Malina（马林纳）发起的火箭研究小组。这个小组在1936年最先在校园里后来转移到离学校不远的San Gabriel山脚下进行过液体小火箭的发射试验，几年以后此地成为美国喷气推进国家实验室的所在地。1937年6—9月，作者 做了有关火箭研究的文献资料的调查研究。在他的一份手稿资料袋中，保存有当年所收集的原始资料以及他所写的调研材料。

原始资料中收集有当年美国的火箭发明家R. H. Goddard在1936年发表在《Scientific American》（科学美国人）杂志上的一篇文章“Liquid - Propellant Rocket Development”，报导Goddard在1930—1935年在新墨西哥州进行的液体火箭飞行试验，1935年的试验中火箭的飞行高度为7500英尺，到达了同温层，该文没有透露有关火箭结构和燃料的技术内容。作者还收集了W. Ley在1935年发表在《Aircraft Engineering》（航空工程）杂志上名为“Rocket Propulsion”（火箭推进）一文的摘录，共9页。该文较全面地讨论了结构材料、实验情况、发动机的冷却、燃料的注人、外围设备等方面的内容。

作者所写的调研材料共有114页。其中的一部分是前人工作的文献目录，分为四个部分，即：A. 早期文献(12篇，1827—1931)，B. 近代书籍

(37本，1913—1933)，C. 近代论文（19篇，1927—1935），及D. 期刊（4种）。调研材料中的大部分则是作者为加州理工学院的火箭研究小组设计和改进小的液体推进剂试验火箭所做的分析计算，内容包括燃烧室中的温度、火箭的理想效率、燃烧产物膨胀不足和过度膨胀对火箭效率的影响、燃料喷咀设计、发动机推力的计算等等。

从这份手稿中可以看到，作者是怎样开始他的火箭研究的。他首先系统地调查、学习和占有前人的经验和知识，然后把课题分解为几个关键问题，在做了某些简化假设以后，利用力学、热力学和化学的基本原理，估算了一些主要控制参数对燃烧效率和推力等的影响。

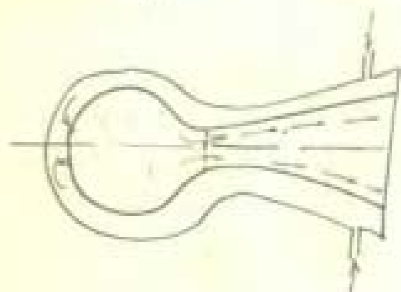
这里我们选印这份手稿中的5页。除了第1页是摘录E. Sänger关于火箭的英国专利说明以外，其他4页取自作者所做的分析计算方面的调研材料：第2页是“火箭发动机的理想循环”一节的首页，第3页是“发动机燃烧室的温度计算”一节的首页，第4页是“火箭的理想效率的计算”一节的首页，第5页是“比热为常数的推力公式”一节的首页。

1

<u>Rocket Engine</u> —— E. Sänger (66, Penzigerstrasse, Vienna, Austria)

British Patent Specification, July 19th, 1935
No. 459,926 ("The Engineer", p. 377, March 27, 1937)

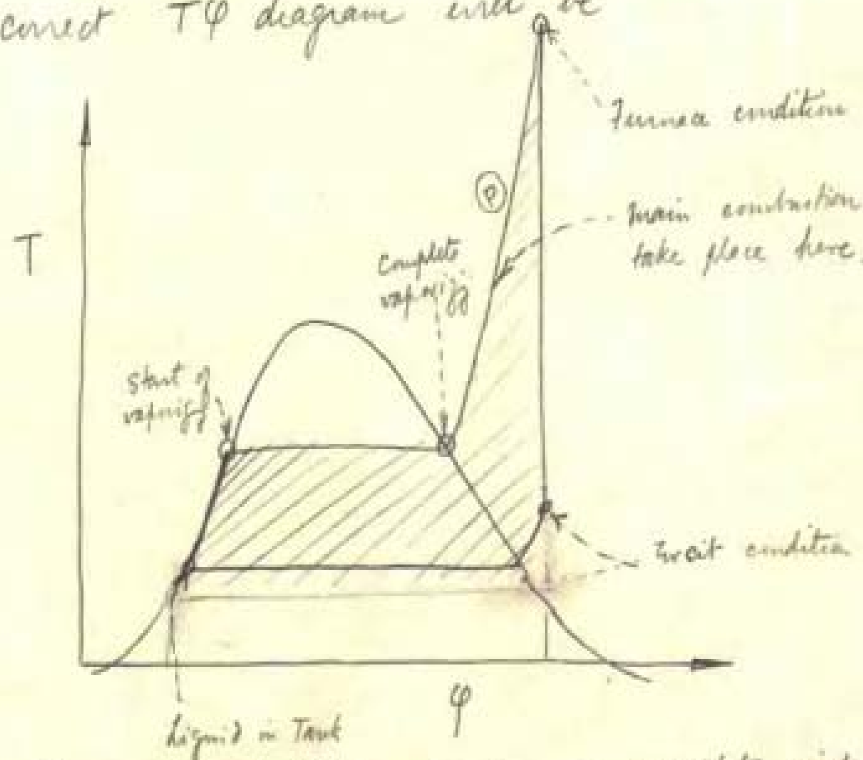
In this specification it is said that for a rocket engine with continuous combustion to have a good efficiency there must be from 50 to 5000 c. cm. of combustion space per cm^2 of cross section of the exhaust outlet, & that the larger ratio is preferable. Oxygen should be used to support combustion & as a consequence very high temperatures are attained. So the combustion chamber is jacketed by coils of piping through which the fuel & oxygen are passed. This results in a preheating of the gases & a cooling of the chamber walls.



Ideal Cycle of Rocket Motor

3 ③

(1) When the combustion material is a liquid, the cycle is very similar to the cycle of a steam power plant. The liquid is first pumped to the heating coil where it is heated to a evaporing temperature under constant pressure + subsequent vaporizing + superheat under constant pressure. Then adiabatic expansion to atmospheric pressure, + then constant pressure cooling to initial liquid state. Therefore the correct Tφ diagram will be



A correct calculation requires a complete information of thermodynamic characteristics of the combustion material

16 ①

Calculation of the Chamber Temperature of a Rocket Motor

Ref: Bulletin No. 139 of Illinois Eng. Exp. Station.

Combustion with Gasoline as Fuel (C_8H_{18})

If there is no preheating of the fuel, then we can take of initial temperature to be the standard temp. of 60°F. or 519.6° F. abs. $= T_1$.

Then the heat of combustion at constant pressure, H_p

$$= 2,145,610 + 519.6\,(-7.218 + 0.8140 \times 0.5196 + 0.1170 \times 0.5196^2)$$

$$= 2,145,610 + 519.6\,(-7.218 + 0.423 + 0.0316)$$

$$= 2,145,610 - 519.6 \times 6.763 = 2,145,610 - 3,512$$

$$= 2,142,100 \text{ B.T.U. per mol of } C_8H_{18}.$$

Consider the reaction



$$C_8H_{18} + 12\tfrac{1}{2}O_2 = 8CO_2 + 9H_2O$$

Therefore if $(1-x_0)$ part of CO_2 is dissociated to $CO + O_2$, $(1-y_0)$ part of H_2O to $H_2 + O_2$, then for $T_1 = 519.6$°F. abs.,

$$H'_{p_{CO}} = 120,930 + 519.6\,(3.245 - 1.95 \times 0.5196 + 0.2600 \times 0.5196^2)$$

$$= 120,930 + 519.6\,(3.245 - 1.012 + 0.0703)$$

$$= 120,930 + 519.6 \times 2.303 = 120,930 + 1,195$$

$$= 122,125 \text{ BTU / mol.}$$

Calculation of Ideal Rocket Efficiency

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Assumptions:

(1) The oxygen + fuel are injected at 60°F. or 519.6°F. ab.

(2) The combustion is at constant pressure

(3) There is no dissociation

(4) There is no heat loss through conduction + radiation

(5) There is no loss by friction

Then the heat content, I, of combustion product

$$= 2,142,100 + 37,400 + 12.5 \times 3,606$$

$$\quad H_p \qquad\qquad H_{48} \qquad\qquad I_{o_2}$$

$$= 2,142,100 + 37,400 + 45,100 = 2,224,600 \text{ BTU.} \; ; \; T_1 = 8,450\text{°F. ab.}$$



$$C_8H_{18} + 12.5\,O_2 \longrightarrow 8CO_2 + 9H_2O \qquad \phi_1 = 1,137.5$$

(I) Chamber pressure = 3000 #/□"

V_2/V_1	$77.75 \log_{10} V_2/V_1$	ϕ_2	T_2	P_2	I_2	$\frac{I_1 - I_2}{I_1}$ %	η_R %	C_v, ft/sec		
500	209.8	927.7	4,120	2,925	750,000	66.3	68.9	11,960		
400	202.2	935.3	4,230	3,757	775,000	65.2	67.8	11,870		
300	192.5	945.0	4,440	5,250	823,000	63.0	65.5	11,660		
200	178.8	958.7	4,760	8.41	900,000	59.6	62.0	11,350		
150	169.2	968.3	4,940	11.69	955,000	57.1	59.3	11,100		
100	155.3	982.2	5,220	18.53	1,038,000	53.4	55.5	10,740		

Thrust Formula for Constant Specific Heats.

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Let S = sectional area of the throat.

Then the mass flow per unit time

$$= S\left(\frac{2k}{k-1}\right)^{\frac{1}{2}}\frac{p_0}{\left(\frac{p_0}{\rho_0}\right)^{\frac{1}{2}}}\left(\frac{p_s}{p_0}\right)^{\frac{1}{k}}\left\{1-\left(\frac{p_s}{p_0}\right)^{\frac{k-1}{k}}\right\}^{\frac{1}{2}}$$

$$= S\left(\frac{2k}{k-1}\right)^{\frac{1}{2}}\frac{p_0}{(RT_0)^{\frac{1}{2}}}\left(\frac{p_s}{p_0}\right)^{\frac{1}{k}}\left\{1-\left(\frac{p_s}{p_0}\right)^{\frac{k-1}{k}}\right\}^{\frac{1}{2}}$$

$$= K_1 S\frac{p_0}{(RT_0)^{\frac{1}{2}}}$$

where $K_1=\left(\frac{2k}{k-1}\right)^{\frac{1}{2}}\left(\frac{p_s}{p_0}\right)^{\frac{1}{k}}\left\{1-\left(\frac{p_s}{p_0}\right)^{\frac{k-1}{k}}\right\}^{\frac{1}{2}}=f_1(k)$

The exit velocity

$$=\left(\frac{2k}{k-1}\right)^{\frac{1}{2}}(RT_0)^{\frac{1}{2}}\cdot\left\{1-\left(\frac{p_e}{p_0}\right)^{\frac{k-1}{k}}\right\}^{\frac{1}{2}}$$

Therefore the thrust of rocket

$$= S\left(\frac{2k}{k-1}\right)p_0\left(\frac{p_s}{p_0}\right)^{\frac{1}{k}}\left\{1-\left(\frac{p_s}{p_0}\right)^{\frac{k-1}{k}}\right\}^{\frac{1}{2}}\left\{1-\left(\frac{p_e}{p_0}\right)^{\frac{k-1}{k}}\right\}^{\frac{1}{2}}$$

$$= K_2\, p_0 S\left\{1-\left(\frac{p_e}{p_0}\right)^{\frac{k-1}{k}}\right\}^{\frac{1}{2}}$$

Where $K_2=\left(\frac{2k}{k-1}\right)\left(\frac{p_s}{p_0}\right)^{\frac{1}{k}}\left\{1-\left(\frac{p_s}{p_0}\right)^{\frac{k-1}{k}}\right\}^{\frac{1}{2}}=f_2(k)$

$$\frac{p_s}{p_0}=\left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}$$

2.2

Flight Analysis of a Sounding Rocket with Special Reference to Propulsion by Successive Impulses

以逐次脉冲推进的探空火箭的飞行分析

这是作者发表于1939年的“Flight Analysis of a Sounding Rocket with Special Reference to Propulsion by Successive Impulses”（以逐次脉冲推进的探空火箭的飞行分析）一文的打字文稿的前两页。这一工作是作者博士论文的一部分。

1936年作者进入美国加州理工学院，追随力学大师 Theodore von Kármán（冯·卡门）学习应用力学。次年，作者参加了学院中由年轻研究生 F. J. Malina（马林纳），A. M. O. Smith（史密斯）等爱好火箭技术而自发形成的研究小组。这个小组的工作特点是实验和理论分析并重，他们自己动手制造模型火箭，进行发射试验；同时对出现的问题和解决方案进行理论分析。作者在这个小组里主要担当理论家的任务。

当时火箭的技术水平是不高的，因此开始设定的目标是研制探空火箭。L. Damblane (1935) 根据静态燃烧试验所做的估计，火箭可达到的高度是10 000英尺，还不能满足探空火箭的需要。这就促使作者做进一步的分析，探讨采用什么样的发动机和燃烧方案可以使火箭经济而高效地达到更高的高度。

作者在这篇文章中探讨和论证了逐次推进的方案，即采用以硝化棉一类的固体火药作为推进剂，进行多次快速燃烧排气而获得脉冲式推力的方案，可以到达离地面100 000英尺的高度，而在这样的高度上就能够进行大气层的结构以及地球大气层以外的物理现象的观测和研究。

PART IV

FLIGHT ANALYSIS OF A SOUNDING ROCKET WITH SPECIAL REFERENCE TO PROPULSION BY SUCCESSIVE IMPULSES

By

Hsue-Shen Tsien and Frank J. Malina
California Institute of Technology

SUMMARY

In Part I of this paper an exact solution of the problem of determining the height reached by a body in vertical flight in vacuo propelled by successive impulses is presented. On the basis of this analysis it is concluded that a rocket propelled by successive impulses - the impulses being obtained, for example, from rapidly burning powder - has useful possibilities and further research is justified. In Part II the effect of the variation of the acceleration of gravity with height above sea level on the flight performance of a sounding rocket is analyzed. In Part III the fundamental performance parameters for flight of a sounding rocket in air are discussed. Finally, in Part IV the theory of the preceding sections is applied to a specific case of a sounding rocket propelled by successive impulses.

INTRODUCTION

In 1919 R. H. Goddard (1) published the historically important paper which suggested the use of nitro-cellulose powder as a propellant for raising a sounding rocket to altitudes beyond the range of sounding balloons. To determine the feasibility of this propellant, a series of experiments had been carried out

-2-

and it was found that a thermal efficiency of 50% could be expected if the powder was exploded in a properly designed chamber and the resulting gases were allowed to escape at high velocity through an expanding nozzle. In 1931 R. Tilling used a mixture of potassium chlorate and naphthalene as propellant and actually reached an altitude of 6,600 feet. More recently, L. Damblanc (Ref. 2) made static tests with a slow burning black powder and from these estimated that a height of 10,000 feet could be reached using a two-step arrangement. The favorable results so far reported offer an incentive to further analysis.

The effect of decreasing gravitational acceleration on the maximum height reached by a rocket has been considered by A. Bartocci (Ref. 3). However, he assumes that the rocket itself has a constant acceleration during powered flight. L. Breguet and R. Devillers (Ref. 4) also considered the effect of the variation of g. To simplify the analysis, they assumed that the acceleration of the rocket was equal to a constant multiple of g. Since the sounding rocket for practical reasons will be propelled by a nearly constant thrust or a uniform rate of successive impulses, in Section II the author has studied the problem anew according to this mode of propulsion.

When the sounding rocket is ascending through the air the maximum height reached is less than that reached for flight in vacuo. Recently, studies have been made of the problem by W. Ley and H. Schaefer (Ref. 5) and by F.J. Malina and A.M.O. Smith (Ref. 6). On the basis of the latter study a group of new performance parameters have been isolated from the general performance equation, and these are discussed in Section III.

2.3

Rockets and Other Thermal Jets Using Nuclear Energy
采用核能的火箭和其他热射流

这是作者发表于1949年的"Rockets and Other Thermal Jets Using Nuclear Energy"（采用核能的火箭和其他热射流）一文的部分手稿，共有6页，即标题为"Nuclear Energy Rocket"的原始手稿的前4页和后2页。

促成第二次世界大战结束的原子弹的使用，激发起人们对于把原子能用于其他工程领域的极大兴趣。作者敏感地意识到，原子能可能用于飞行器的动力装置，因为它能适应超声速飞行所要求的降低燃料重量和增加有效负载这两个主要指标。

作者在战后不久，即1946年，在《Journal of Aeronautical Sciences》（航空学报）上发表了"Atomic Energy"（原子能）一文，介绍了用原子能作为飞行器动力这一新的研究领域所需要的基础知识。同时，在《Nuclear Science and Engineering》（核科学和核工程）一书的第二卷中写了题为"Rockets and Other Thermal Jets Using Nuclear Energy"（用核能的火箭和其它热射流）的一章，这里发表的就是作者写作这一章的部分手稿。文中讨论了采用核动力的火箭及其他喷气推进装置中出现的几个基本问题，如：相对论效应，优化设计等；接着，作者对核动力火箭的性能和重量做了估算，并就减小临界尺寸的可能性与采用多孔材料作为堆体的优点等问题提出了建议。现在看来，作者的上述观点确实抓住了核航天技术的一些关键。

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Nuclear Energy Rocket

$N(238)/N(235) = 1$.

Moderator, Be

Let $N(Be)/N(238) = R$

Density of alloy $= 1.85$ gr./cm.^{3}

Average atomic weight of metal $= \dfrac{238 + 235 + 9.02R}{R+2}$

No. of atoms per cm^3 of metal

$$= 1.85 \times 6.06 \times 10^{23} \frac{R+2}{473 + 9.02R}$$

If ϑ is the porosity of the structure, then

No. of metallic atoms per cm^3 of pile

$$= (1-\vartheta)\, 1.85 \times 6.06 \times 10^{23} \frac{R+2}{473 + 9.02R}$$

Under standard conditions, the density of H_2 gas is 0.0000899 gr/cm^3.

At 300 psi $= \dfrac{300}{14.7} = 20.4$ atm.

At 2730° K., Density $= 2.04 \times 0.0000899$ gr./cm^3

$= 0.0001833$ gr./cm^3

No. of H-atoms per cm^3 of pile

$$= \vartheta\, 0.0001833 \times 6.06 \times 10^{23}$$

2

$$N(235) = 1.85 \times 6.06 \times 10^{22} (1-\vartheta) \frac{1}{473+9.02R}$$

$$N(238) = 1.85 \times 6.06 \times 10^{22} (1-\vartheta) \frac{1}{473+9.02R}$$

$$N(Be) = 1.85 \times 6.06 \times 10^{22} (1-\vartheta) \frac{R}{473+9.02R}$$

$$N(H) = 0.0001833 \times 6.06 \times 10^{22}\, \vartheta$$

Therefore

$$f = \frac{\sigma_f(U235)N(235)}{\sigma_f(U235)N(235) + \sigma_c(U238)N(238) + \sigma_c(Be)N(Be) + \sigma_c(H)N(H)}$$

$$= \frac{1.85(1-\vartheta)\frac{500}{473+9.02R}}{1.85(1-\vartheta)\frac{500}{473+9.02R} + 1.85(1-\vartheta)\frac{2}{473+9.02R} + 1.85(1-\vartheta)\frac{0.009R}{473+9.02R} + 0.0001833\vartheta \times 0.31}$$

$$\boxed{f = \frac{1.85 \times 500 (1-\vartheta)}{1.85(1-\vartheta)(502 + 0.009R) + 0.0001833 \times 0.31\, \vartheta\, (473+9.02R)}}$$

$$\int \frac{\sigma_a(E)N_a}{\sigma_a(E)N_a + \sigma_s(E)N_s} \frac{dE}{\xi E}$$

$$\cong \int \frac{\sigma_a(E)\,dE}{E} \frac{1}{\xi} \frac{N_a(238)}{\sigma_s(Be)N(Be) + \sigma_s(H)N(H)}$$

$$= 88\, \frac{1}{\xi} \frac{N(238)}{\sigma_s(Be)N(Be) + \sigma_s(H)N(H)}$$

3

$$\xi(Be) = 0.206$$

$$\xi(H) = 1$$

$$\xi = \frac{\xi(H)\sigma_s(H)N(H) + \xi(Be)\sigma_s(Be)N(Be)}{\sigma_s(H)N(H) + \sigma_s(Be)N(Be)}$$

$$\int \frac{\sigma_R(E)N_2}{\sigma_R(E)N_2 + \sigma_s(E)N_1}\frac{dE}{\xi E} = 88\frac{N(238)}{1\cdot\sigma_s(H)N(H) + 0.206\,\sigma_s(Be)N(Be)}$$

$$p = e^{-\frac{1.85(1-\vartheta)\times 88}{20\times 0.0001103\,\vartheta(473+9.02R) + 0.206\times 3.8\times 1.85(1-\vartheta)R}}$$

$$k = 2.3\,pf$$

$$\frac{\partial k}{\partial R} = 0;$$

$$\frac{1.85(1-\vartheta)\times 88\left[0.003666\,\vartheta\times 9.02 + 0.206\times 3.8\times 1.85(1-\vartheta)\right]}{\left\{0.003666\,\vartheta(473+9.02R) + 0.206\times 3.8\times 1.85(1-\vartheta)R\right\}^2}$$

$$-\frac{1.85(1-\vartheta)\times 0.009 + 0.0001833\times 0.31\,\vartheta\times 9.02}{1.85(1-\vartheta)(502+0.009R) + 0.0001833\times 0.31\,\vartheta(473+9.02R)} = 0.$$

4

$$162.7(1-\vartheta)\left[0.03306\,\vartheta+1.448(1-\vartheta)\right]\left[928(1-\vartheta)+0.0269\,\vartheta+\left\{0.01665(1-\vartheta)+0.000512\,\vartheta\right\}R\right]$$

$$-\left\{0.01665(1-\vartheta)+0.000512\,\vartheta\right\}\left[1.732\,\vartheta+\left\{0.03306\,\vartheta+1.448(1-\vartheta)\right\}R\right]^2=0.$$

Or let $\vartheta=0.7$ (porous metal part is actually 60% metal!)

$$162.7\times0.3\times\left[0.02312+0.435\right]\left[278.4+\left\{0.005+0.00036\right\}R\right]$$

$$-\left\{0.005+0.00036\right\}\left[1.211+\left\{0.02312+0.435\right\}R\right]^2=0.$$

$$162.7\times0.3\times0.458\times(278.4+0.00536R)-0.00536(1.211+0.458R)^2=0.$$

$$4170(278.4+0.00536R)-1.211^2-1.211\times0.916R-0.458^2R^2=0.$$

$$0.458^2R^2-(22.36-1.11)R-(4170\times278.4+1.211^2)=0.$$

Let $R\cong2000.\quad\vartheta=0.7$

$$f=\frac{1.85\times150}{1.85\times0.3(502+18)+0.1833\times0.3/0.7\times18.51}$$

$$=\frac{150}{156+0.398}=0.960$$

$$\int\frac{\sigma_a(E)N_a}{\sigma_a(E)N_a+\sigma_s(E)N_s}\frac{dE}{\xi E}=\frac{1.85\times0.3\times28}{0.003666\times0.7\times10^4\times18.51+0.206\times3.8\times1.85\times0.3\times2000}$$

$$=\frac{25.4}{36.66\times0.7+0.206\times3.8\times600}=\frac{25.4}{25.62+470}=[illegible]$$

28

Weight flow = 452 lb/sec.

Thrust = 170 tons.

Summary of Results

Porous Tube to be made of a mixture of U-235, U-238 and C.

Wall thickness of tube = $\frac{1}{8}$"

Porosity of the tube material = 0.40

Tube to be uniformly tapered; average dimension as follows:



Approximately double size

Temperature differential of porous material and the gas = 17°F,

assuming equivalent capillary tubes of 0.00759" diameter.

Pressure drop across the porous material ≅ 1 psi

Chamber pressure = 300 psi

Final Chamber temperature = 6000°R = 3333°K = 3060°C.

Exhaust velocity ≅ 24,000 ft/sec.

	Example I $N(235)/N(238) = \frac{1}{10}$	Example II $N(235)/N(238) = 1$
$N(C)/N(235)$	1070	2260
Diameter of active mass, inches	206	166
Length of active mass, inches	103	83
Weight of active mass, tons	41.9	22.04
Weight of U-235, lbs.	149.8	256.8
Weight flow of H_2, lbs/sec.	8,730	4570
Thrust, tons	1,260	1,705
Energy rate, B.t.u./sec.	1.874×10^8	0.987×10^8
Kw	1.987×10^8	1.044×10^8
Weight of fuel, 48% burnt, tons	1,582	818
Duration of burning, sec.	358	358
Total amount of heat used, B.t.u.	6.75×10^{13}	3.53×10^{13}

3×10^{10} fissions per second ≅ 1 watt.

$3 \times 10^{10} \times 1000 \times 3600$ = kw-hr = 3413 B.t.u.

$$1 \text{ B.t.u.} = 3 \times 10^{13} \times \frac{3600}{3413} \text{ fissions}$$

1 lb. of U-235 = 453 grams = 1.93 gr-mole = $1.93 \times 0.606 \times 10^{24}$ atoms

$$= \frac{1.93 \times 0.606 \times 10^{24}}{3 \times 10^{13} \times 1.054} = 3.700 \times 10^{10} \text{ B.t.u.}$$

2.4

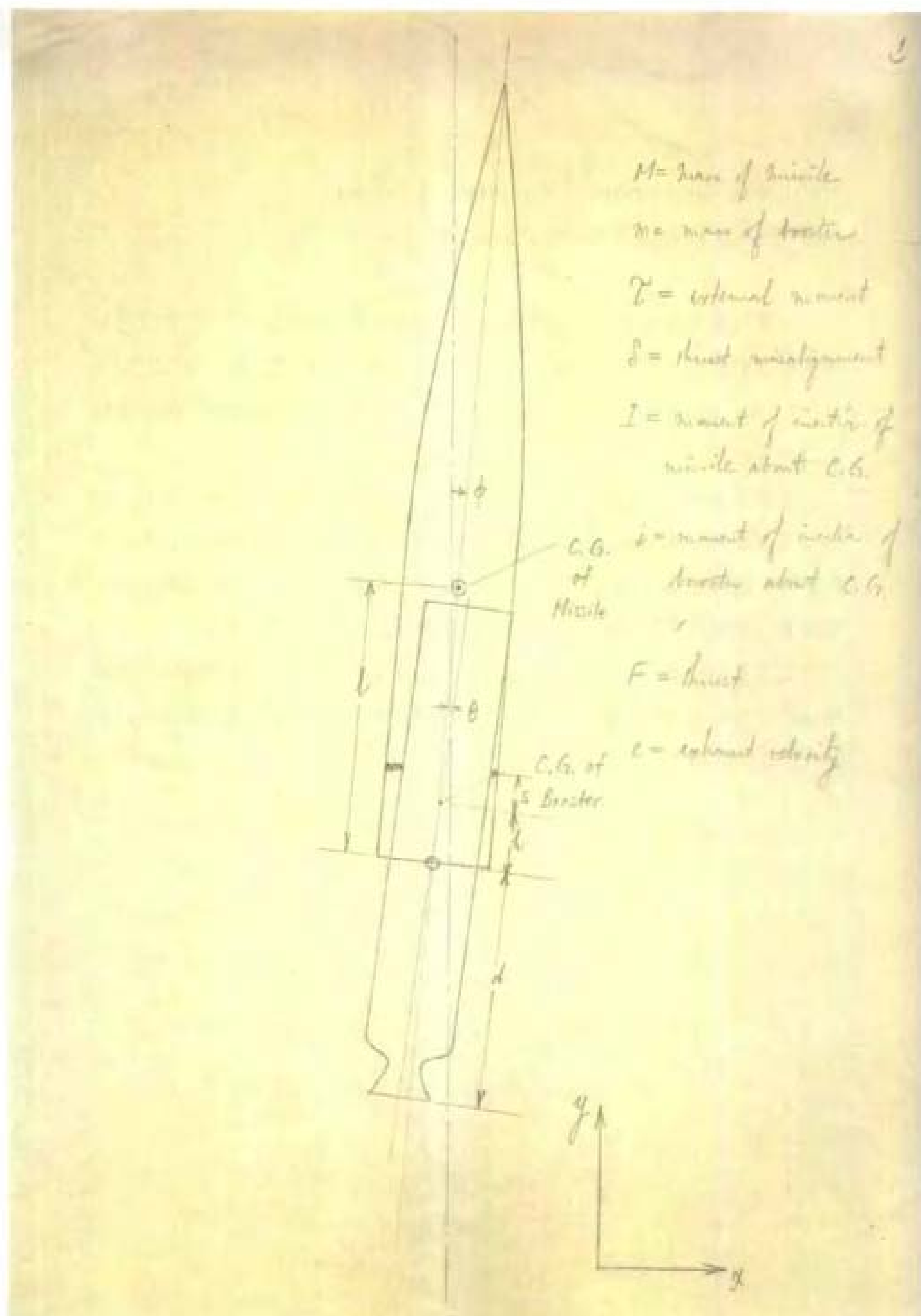
Stabilizing by Dynamic Mounting of Booster

火箭助推器动态支撑的稳定作用

这是作者未曾发表的一份手稿中的一部分。在这份手稿中，作者设想利用助推器的动态支撑而实现火箭的稳定飞行，并且进行了分析，写作时间不详。F. Marble（弗朗克·马勃）认为，这个问题是作者建议加州理工学院的研究生进行研究的课题。

当助推器的推力偏离火箭飞行的方向时，可能导致两种不同的后果。一种是飞行器更加偏离原轨道，这时系统是不稳定的；另一种则是回到原定方向上来，这时系统是稳定的。稳定与否取决于助推器与飞行器间支撑的动力学参数，参数选择适当就可以得到稳定的系统，实现稳定的飞行。

这里选印作者第一份底稿，共 10 页。前 8 页是推导由火箭及其助推器构成的系统的运动方程组，以及与之相应的反映系统稳定性的特征方程，后 2 页是数值算例。


M = mass of missile
m = mass of booster
$\mathcal{T}$ = external moment
δ = thrust misalignment
I = moment of inertia of missile about C.G.
i = moment of inertia of booster about C.G.
F = thrust
c = exhaust velocity
C.G. of Missile
C.G. of Booster
y
x

2

For the linear acceleration, we shall use a rectangular coordinate system fixed in space:

For the missile:

$$F - Mg = M\ddot{y} \qquad (1)$$

$$F(\theta+\phi) + \frac{E}{c}\left[d(\dot{\theta}+\dot{\phi}) + l\dot{\phi}\right] = M\ddot{x} \qquad (2)$$

For angular acceleration, we shall use a system passing through the C.G. of the missile:

$$I\ddot{\phi} - \frac{E}{c}\left(\frac{J}{M}\right)\dot{\phi} + \frac{E}{c}(l+d)\left[d(\dot{\theta}+\dot{\phi}) + l\dot{\phi}\right] + Fl\cdot\theta = \tau + F\delta$$

Or

$$\boxed{I\ddot{\phi} + \frac{E}{c}\left[(l+d)^2 - \frac{J}{M}\right]\dot{\phi} + \frac{E}{c}d(l+d)\dot{\theta} + Fl\theta = \tau + F\delta} \qquad (3)$$

For the booster

$$F + R_y - mg = m\ddot{y} \qquad (4)$$

$$\frac{E}{c}(h+d)(\dot{\theta}+\dot{\phi}) + F(\theta+\phi) + R_x - (l+h)\theta\alpha = m\left[\ddot{x} - l\ddot{\phi} + h(\ddot{\theta}+\ddot{\phi})\right] \qquad (5)$$

For angular acceleration, we again use a system passing C.G. of booster:

$$i(\ddot{\theta}+\ddot{\phi})+\frac{E}{c}(h+d)^2(\dot{\theta}+\dot{\phi})-\frac{E}{c}\left(\frac{\dot{i}}{m}\right)(\dot{\phi}+\dot{\theta})+R_x h-R_y h(\theta+\phi)+(s+h)\theta\alpha\alpha=F\delta \qquad (6)$$

From (1) and (4)

$$R_y=m\ddot{y}+mg-F=m(\ddot{y}+g)-F$$

$$\therefore\ R_y=F\left(\frac{m}{M}-1\right) \qquad (7)$$

From (2) and (5)

$$R_x=m\left[\frac{F(\theta+\phi)+\frac{E}{c}\{d(\dot{\theta}+\dot{\phi})+l\dot{\phi}\}}{M}-l\ddot{\phi}+h(\ddot{\theta}+\ddot{\phi})\right]$$

$$-\frac{E}{c}(h+d)(\dot{\theta}+\dot{\phi})-F(\theta+\phi)+(s+h)\theta\alpha \qquad (8)$$

Substituting (7) and (8) into (6),

$$(mh^2+i)(\ddot{\theta}+\ddot{\phi})+\frac{E}{c}d(h+d)^2(\dot{\theta}+\dot{\phi})-\frac{E}{c}\left(\frac{\dot{i}}{m}\right)(\dot{\phi}+\dot{\theta})+$$

$$+mh\left[\frac{F(\theta+\phi)+\frac{E}{c}\{d(\dot{\theta}+\dot{\phi})+l\dot{\phi}\}}{M}-l\ddot{\phi}+h(\ddot{\theta}+\ddot{\phi})\right]$$

$$-\frac{E}{c}h(h+d)(\dot{\theta}+\dot{\phi})-Fh(\theta+\phi)+(s+h)^2\theta\alpha$$

$$-Fh\left(\frac{m}{M}-1\right)(\theta+\phi)+s(s+h)\theta\alpha=F\delta$$

4

$$(i+mh^2)(\ddot\theta+\ddot\phi)-mhl\ddot\phi+\frac{F}{c}\left[d(d+h)-\frac{i}{m}+\frac{m}{M}hd\right](\dot\theta+\dot\phi)+\frac{m}{M}hl\dot\phi$$

$$+\beta\theta=FS$$

$$\boxed{\beta=\alpha(s+h)^2} \quad \text{Spring constant} \qquad (9)$$

$$\boxed{\begin{aligned}&\{i-mh(l-h)\}\ddot\phi+\frac{F}{c}\left[d(d+h)-\frac{i}{m}+\frac{m}{M}h(d+l)\right]\dot\phi\\ &\qquad +(i+mh^2)\ddot\theta+\frac{F}{c}\left[d(d+h)-\frac{i}{m}+\frac{m}{M}hd\right]\dot\theta+\beta\theta=FS\end{aligned}} \qquad (10)$$

Let $\quad I=Mk^2$

$\quad l^*=l/k, \text{ etc.}$

(3) becomes

$$\ddot\phi+\frac{F}{cM}\left[(l^*+d^*)^2-1\right]\dot\phi+\frac{F}{cM}d^*(l^*+d^*)\dot\theta+\frac{Fl}{I}\theta=\frac{?}{I}+\frac{FS}{I}$$

(10) becomes

$$\left\{\left(\frac{i}{I}\right)-\left(\frac{m}{M}\right)h^*(l^*-h^*)\right\}\ddot\phi+\frac{F}{cM}\left[d^*(d^*+h^*)-\left(\frac{i}{I}\right)\left(\frac{M}{m}\right)+\left(\frac{m}{M}\right)h^*(d^*+l^*)\right]\dot\phi$$

$$+\left\{\left(\frac{i}{I}\right)+\frac{m}{M}h^{*2}\right\}\ddot\theta+\frac{F}{cM}\left[d^*(d^*+h^*)-\left(\frac{i}{I}\right)\left(\frac{M}{m}\right)+\frac{m}{M}h^*d^*\right]\dot\theta+\frac{\beta}{I}\theta$$

$$=\frac{FS}{I}$$

5

Or

$$\left\{1-\left(\frac{m}{M}\right)\left(\frac{I}{i}\right)k^*(l^*-k^*)\right\}\ddot{\phi}+\left(\frac{F}{cM}\frac{I}{i}\right)\left[d^*(d^*+k^*)-\left(\frac{i}{I}\right)\left(\frac{M}{m}\right)+\left(\frac{m}{M}\right)k^*(d^*+l^*)\right]\dot{\phi}$$

$$+\left\{1+\frac{m}{M}\frac{I}{i}k^{*2}\right\}\ddot{\theta}+\left(\frac{F}{cM}\frac{I}{i}\right)\left[d^*(d^*+k^*)-\left(\frac{i}{I}\right)\left(\frac{M}{m}\right)+\frac{m}{M}k^*d^*\right]\dot{\theta}+\frac{\beta}{i}\theta$$

$$=\frac{F\delta}{i}$$

Let

$$\left.\begin{aligned} t &= t^*\sqrt{\frac{I}{Fk}} \\ t^* &= t\sqrt{\frac{Fk}{I}} \\ \phi' &= \frac{d\phi}{dt^*} \qquad L = l^*+d^*, \quad D = k^*+d^* \end{aligned}\right\} \quad (11)$$

$$\phi''+\left(\frac{F}{cM}\sqrt{\frac{I}{Fk}}\right)(L^2-1)\phi'+\left(\frac{F}{cM}\sqrt{\frac{I}{Fk}}\right)L(L-l^*)\theta'+l^*\theta=\frac{\tau}{Fk}+\frac{\delta}{k}$$

$$\left\{1-\left(\frac{m}{M}\right)\left(\frac{I}{i}\right)(L-D)[l^*-(L-D)]\right\}\phi''+\left(\frac{F}{cM}\sqrt{\frac{I}{Fk}}\frac{I}{i}\right)\left[D(L-l^*)-\left(\frac{i}{I}\right)\left(\frac{M}{m}\right)+\frac{m}{M}L\{l^*-(L-D)\}\right]\phi'$$

$$+\left\{1+\frac{m}{M}\frac{I}{i}[l^*-(L-D)]^2\right\}\theta''+\left(\frac{F}{cM}\sqrt{\frac{I}{Fk}}\frac{I}{i}\right)\left[D(L-l^*)-\left(\frac{i}{I}\right)\left(\frac{M}{m}\right)+\frac{m}{M}(L-l^*)\{l^*-(L-D)\}\right]\theta'$$

$$+\left(\frac{\beta}{i}\frac{I}{Fk}\right)\theta=\frac{F\delta}{i}\frac{I}{Fk}$$

6

Let

$$\frac{F}{CM}\sqrt{\frac{I}{Fk}} = z, \quad \frac{\beta}{Fk} = n^2$$

$$\phi'' + z(L^2-1)\phi' + zL(L-\ell^*)\theta' + \ell^*\theta = \frac{\gamma}{Fk} + \frac{\delta}{k} \tag{12}$$

$$\left[\left(\frac{i}{I}\right) - \left(\frac{m}{M}\right)(L-D)\{\ell^* - (L-D)\}\right]\phi'' + z\left[D(L-\ell^*) - \left(\frac{i}{I}\right)\left(\frac{M}{m}\right) + \frac{m}{M}L\{\ell^* - (L-D)\}\right]\phi'$$

$$+\left[\left(\frac{i}{I}\right) + \frac{m}{M}\{\ell^* - (L-D)\}^2\right]\theta'' + z\left[D(L-\ell^*) - \left(\frac{i}{I}\right)\left(\frac{M}{m}\right) + \frac{m}{M}(L-\ell^*)\{\ell^* - (L-D)\}\right]\theta' \tag{13}$$

$$+ n^2\theta = \frac{\delta}{k}$$

For a given design, $L, D, z, \left(\frac{i}{I}\right), \left(\frac{M}{m}\right)$ fixed;

ℓ^*, n^2 can be varied.

$l_1 = L - D$ = distance between C.G. of missile to C.G. of booster in units of k.

7

$$\left[(\ )'' + 2(L^2-1)(\ )'\right]\phi + \left[2L(L-l^2)(\ )' + l^2\ \right]\theta = \frac{2}{Fh} + \frac{s}{h}$$

$$\left[\left\{\left(\frac{i}{2}\right) - \left(\frac{m}{M}\right)l_1(l^2-l_1)\right\}(\ )'' + 2\left\{D(L-l^2) - \left(\frac{i}{2}\right)\left(\frac{M}{m}\right) + \frac{m}{M}L(l^2-l_1)\right\}(\ )'\right]\phi$$

$$+\left[\left\{\left(\frac{i}{2}\right) + \left(\frac{m}{M}\right)(l^2-l_1)^2\right\}(\ )'' + 2\left\{D(L-l^2) - \left(\frac{i}{2}\right)\left(\frac{M}{m}\right) + \frac{m}{M}(L-l^2)(l^2-l_1)\right\}(\ )' + n^2\right]\theta = \frac{s}{h}$$

The characteristic equation:

$$\boxed{a\lambda^4 + b\lambda^3 + c\lambda^2 + d\lambda = 0}$$

$$\boxed{a = \left(\frac{i}{2}\right) + \left(\frac{m}{M}\right)(l^2-l_1)^2}$$

$$b = 2\left\{D(L-l^2) - \left(\frac{i}{2}\right)\left(\frac{M}{m}\right) + \frac{m}{M}(L-l^2)(l^2-l_1)\right\} + 2(L^2-1)\left\{\left(\frac{i}{2}\right) + \frac{m}{M}(l^2-l_1)^2\right\}$$

$$- 2L(L-l^2)\left\{\left(\frac{i}{2}\right) - \frac{m}{M}l_1(l^2-l_1)\right\}$$

$$\boxed{b = 2\left[D(L-l^2) - \left(\frac{i}{2}\right)\left(\frac{M}{m}\right) + \left(\frac{i}{2}\right)(Ll^2-1) + \frac{m}{M}(l^2-l_1)\left\{l^2(L^2-1-Ll_1) + l_1\right\}\right]}$$

8

$$c = n^2 + z^2(L^2-1)\left\{D(L-\ell^*) - \left(\frac{i}{2}\right)\left(\frac{M}{m}\right) + \frac{m}{M}(L-\ell^*)(\ell^*-\ell_1)\right\}$$

$$- \ell^*\left\{\left(\frac{i}{2}\right) - \left(\frac{m}{M}\right)\ell_1(\ell^*-\ell_1)\right\} - z^2 L(L-\ell^*)\left\{D(L-\ell^*) - \left(\frac{i}{2}\right)\left(\frac{M}{m}\right) + \frac{m}{M}L(\ell^*-\ell_1)\right\}$$

$$c = n^2 - \ell^*\left\{\left(\frac{i}{2}\right) - \left(\frac{m}{M}\right)\ell_1(\ell^*-\ell_1)\right\} + z^2\left[(L\ell^*-1)\left\{D(L-\ell^*) - \left(\frac{i}{2}\right)\left(\frac{M}{m}\right) + \frac{m}{M}L(\ell^*-\ell_1)\right\} - (L^2-1)\frac{m}{M}\ell^*(\ell^*-\ell_1)\right]$$

$$d = z\left[n^2(L^2-1) - \ell^*\left\{D(L-\ell^*) - \left(\frac{i}{2}\right)\left(\frac{M}{m}\right) + \frac{m}{M}L(\ell^*-\ell_1)\right\}\right]$$

9

Typical Example

$\frac{m}{M} = \frac{1}{3}$

$\frac{i}{I} = \frac{1}{27}$

$l_1 = 1$

$l_2 = 1.7$

$D = 0.7$

$\varepsilon = 0.02$

$$a = \frac{1}{27} + \frac{1}{3}(l^2-1)^2 > 0$$

$$b = 0.02\left[0.7(1.7 - l^4) - \frac{1}{7} + \frac{1}{27}(1.7l^2-1) + \frac{1}{3}(l^2-1)\{0.19l^2+1\}\right]$$

$$= 0.02\left[1.19 - 0.7l^4 - \frac{1}{7} + \frac{1.7}{27}l^2 - \frac{1}{27} + \frac{1}{3}(0.19l^4 + 0.81l^2 - 1)\right]$$

$$b = 0.02\left[\frac{0.19}{3}l^4 - \left(0.7 - \frac{1.7}{27} - 0.27\right)l^2 + 1.19 - \frac{1}{7} - \frac{1}{27} - \frac{1}{3}\right]$$

$b = 0.02 \times \frac{1}{27}\left[1.71 l^4 - 7.71 l^2 + 19.13\right]$	$b > 0$

$$c = n^2 - l^4\left\{\frac{1}{27} - \frac{1}{3}(l^2-1)\right\} + 0.02^2\left[(1.7l^2-1)\left\{0.7(1.7-l^4) - \frac{1}{7} + \frac{1}{3}1.7(l^2-1)\right\} - 1.19\frac{1}{3}l^4(l^2-1)\right]$$

10

$$c = n^2 + \frac{l^4}{3}\left\{(l^4-1) - \frac{1}{9}\right\} + 0.02^2\left[(1.7l^4-1)\left\{1.17 - 0.7l^4 - \frac{1}{9} + \frac{1.7}{3}l^4 - \frac{1.7}{3}\right\} - 0.63\,l^4(l^4-1)\right]$$

$$= n^2 + \frac{l^4}{3}\left\{(l^4-1) - \frac{1}{9}\right\} + 0.02^2\left[(1.7l^4-1)\left(-\frac{0.4}{3}l^4 + \frac{4.31}{9}\right) - 0.63\,l^4(l^4-1)\right]$$

$$= n^2 + \frac{l^4}{3}\left\{(l^4-1) - \frac{1}{9}\right\} + 0.02^2\left[-\frac{0.68}{3}l^{42} + \frac{4.31\times1.7}{9}l^4 + \frac{0.4}{3}l^4 - \frac{4.31}{9} - 0.63\,l^{42} + 0.63\,l^4\right]$$

$$\boxed{c = n^2 + \frac{l^4}{3}\left\{(l^4-1) - \frac{1}{9}\right\} - 0.02^2\left[+\frac{2.57}{3}l^{42} - \frac{14.197}{9}l^4 + \frac{4.31}{9}\right]}$$

$$\boxed{d = 0.02\left[1.89n^2 + l^4\left(+\frac{0.4}{3}l^4 - \frac{4.31}{9}\right)\right]}$$

$$bc > ad$$

Let $l^4 = 1.3$

$$d = 0.02\left[1.89n^2 - 1.3\left(\frac{0.26}{3} + \frac{1.75}{9}\right)\right] = 0.02\left[1.89n^2 - \frac{1.3\times2.33}{9}\right]$$

$$n^2 > \frac{1.3\times2.33}{9\times1.89} = \frac{1.603}{9} = 0.1781$$

2.5

Long Range Commercial Rocket

远程商用火箭

作者曾在美国 AIAA 年会上做过题为" Long Range Commercial Rocket "（远程商用火箭）的报告，在社会上引起很大的反响（见当时的美国的《时代杂志》）。

1944 年，作者参加了加州理工学院喷气推进实验室所接受的研制远程火箭的一个大型研究项目的工作。为了制订远程火箭的研究路线和方案，作者对当时各种类型的火箭发动机的优缺点作了比较性分析，因此熟悉了发动机的性能。与此同时，作者对于将火箭喷气推进技术推广到商业应用的可能性发生了兴趣。

作者注意了解当时有关这一问题的学术动态，学者们对喷气运输机的经济性已经做过仔细的研讨，认为：无论是亚声速或超声速的喷气运输机，高速飞行所需的成本太高。作者仔细地分析了上述结论的依据，发现前人的结论基于传统的飞行路线，即在一定的高度上做等速飞行。如果改用另一种飞行路线和方式，有可能解决经济性的问题。

作者的新方案的要点是：采用高推力的火箭发动机，在相对短的时间内产生足够的动能，使飞机垂直向上起飞冲出大气层；然后在熄火无动力的状态下，飞机在无空气阻力的高真空中沿椭圆形轨道飞行；当其重新进入大气层后，再可利用机翼所接受的空气动力的作用，使飞机在相当长的一段距离内做小角度的俯冲滑行而直达目的地。这一方案可以显著减少燃料消耗而增大航程。

1948 年，作者以 Aerojet Engineering Corporation 公司顾问的身份，将一份题为" MEMORANDUM on : Optimum Trajectory for Long Range Rocket Missiles"（关于远程火箭导弹的优化轨道的备忘录）的研究报告送交该公司的副总裁 D. A. Kimball。作者对方案中多种可能选择的滑行轨道作了估

算，为探求达到最大航程的优化轨道，做了一个全局性的论证。该报告的手稿一共 25 页，这里选印手稿的前 10 页以及一页有关滑行轨道分析的内容目录。而在前 10 页中，最前面的 3 页写出了作者分析这一问题的总思路，包括基本假设、分析方法和主要结果。后面 7 页讨论的是下面这样一个无升力的滑行轨道的优化问题：当最初的依靠火箭动力的加速阶段结束以后，飞机冲出了大气层，在大气层外开始做无升力的滑行。如果滑行的初速为 V_0，航向与地面的夹角为 ψ，那么要问：对于一个给定的初速 V_0，采取什么样的方向角 ψ 可以达到最大的航程？作者给出了这一优化问题的解答。

过了几年，作者又指导他的研究生 D. D. Beyer 在上述方案分析的基础上，就方案的经济性进行了论证，在 1953 年写出了题为 " Economic Possibilities of Long Range Commercial Rocket Transports " （远程商用火箭运输的经济上的可能性）的学位论文。

4 ab-S 1

Memorandum on: Optimum Trajectory for Long Range Rocket Missiles

To: Mr. D. A. Kimball, Vice President, Aerojet Engineering Corporation

From: H. S. Tsien, Consultant

Date: Jan 11, 1948

Content! (see next page)

1. Assumptions and Summary of Results

With a given fraction of the gross weight of a rocket missile taken as the propellant weight, there is always the problem of optimum trajectory for maximum range. To solve this problem accurately is extremely tedious and laborious. Therefore before attacking this problem accurately, some preliminary calculations designed to obtain the rough outline of the solution are instructive. For these calculations which will be explained in the following sections, the following simplifying assumptions are made:

a) The atmosphere is assumed to be concentrated in a relatively thin layer near the surface of the earth;

b) The flight of the missile without lifting wings is high above this layer of atmosphere and is thus without air drag

c) The gliding flight of the missile with lifting wings, must be then in this thin layer of atmosphere and thus follows relatively closely the surface of the earth

d) The trajectory of the missile can be turned away from the surface of earth by the action of lifting wing but then this must occur near the surface of the earth where the atmosphere is

e) The rotation of earth is neglected, but the curvature of earth is taken into account.

The analysis gives the following <u>main results</u>:

a) With the same fraction of gross weight as propellant, the range of the missile can be increased by roughly 100% with lifting wing. <u>After a initial powered accelerating period,</u>

b) The wing can be used to obtain a multi-loop trajectory, or a steady glide trajectory, or one loop and then steady glide ~~trajectory~~. The range of these trajectories is increasing in the order mentioned. However the difference among them is not very large.

c) Steady flight at constant speed with wing and rocket power is inefficient in obtaining range. This type of trajectory is definitely undesirable.

2. <u>Method of Analysis</u>

The trajectory is subdivided into parts of five different types which can be joined to form a wide variety of trajectory:

a) Coasting flight without lift
b) Coasting flight around the earth with lift
c) Coasting flight in an upward circular path with lift
d) Powered flight around the earth at constant velocity with lift.
e) Initial powered flight to obtain required velocity.

These particular parts of trajectory will be now analyzed:

a) <u>Coasting flight without lift</u>

The flight is considered to be started at the surface of the

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earth with a velocity v_0 which is inclined to the surface by an angle ψ (Fig. 1). Use the polar coördinates (r, θ) and let $\theta = 0$ corresponds to the summit of the trajectory. $\theta = -\theta_0$ is the value of θ at the starting point of the flight. Due to symmetry of the trajectory, the range s_R is

$$s_R = 2R\theta_0 \tag{1}$$

where R is the radius of the earth,

$$R = 20.9 \times 10^6 \text{ ft.} \tag{2}$$

Let the mass of the missile be m, then the force of attraction from the center of the earth is

$$\frac{km}{r^2}$$

where k is a constant. The potential energy at r due to this force of attraction is equal to the negative of the work done in letting the mass "fall" from infinity to r. Thus

$$\text{potential energy} = -\int_\infty^r \frac{km}{r^2}(-dr) = km\left[-\frac{1}{r}\right]_\infty^r = -\frac{km}{r} \tag{3}$$

The kinetic energy is given by

$$\text{kinetic energy} = \frac{1}{2}m\left[\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2\right] \tag{4}$$

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Since the air drag is assumed to be zero, the sum of the potential energy and the kinetic energy must be a constant equal to the value at the starting point. Hence

$$\frac{1}{2}\left[\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2\right] - \frac{k}{r} = \frac{1}{2}V_0^2 - \frac{k}{R} \qquad (5)$$

The linear momentum in the circumferential direction is $mr\frac{d\theta}{dt}$. Therefore the moment of momentum is $mr^2\frac{d\theta}{dt}$. Now since the central attractive force will not act on this quantity, the moment of momentum will be constant and equal to the value at the starting point. Hence

$$r^2\frac{d\theta}{dt} = RV_0\cos\psi \qquad (6)$$

Now

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt}$$

By (6), then

$$\frac{dr}{dt} = \frac{dr}{d\theta}\left[\frac{RV_0\cos\psi}{r^2}\right] \qquad (7)$$

By substituting (7) into (5),

$$\left(\frac{dr}{d\theta}\right)^2\frac{R^2V_0^2\cos^2\psi}{r^4} + \frac{R^2V_0^2\cos^2\psi}{r^2} - \frac{2k}{r} = V_0^2 - \frac{2k}{R} \qquad (8)$$

Therefore

$$\left(\frac{dr}{d\theta}\right)\frac{RV_0\cos\psi}{r^2} = \sqrt{V_0^2 - \frac{2k}{R} - \frac{R^2V_0^2\cos^2\psi}{r^2} + \frac{2k}{r}} \qquad (9)$$

By separating the variables,

$$d\theta = \frac{-d\left(\frac{RV_0\cos\psi}{r}\right)}{\sqrt{V_0^2 - \frac{2k}{R} - \frac{R^2V_0^2\cos^2\psi}{r^2} + \frac{2k}{r}}}$$

$$= \frac{-d\left\{\frac{RV_0\cos\psi}{r} - \frac{k}{RV_0\cos\psi}\right\}}{\sqrt{\left\{V_0^2 - \frac{2k}{R} + \frac{k^2}{R^2V_0^2\cos^2\psi}\right\} - \left\{\frac{RV_0\cos\psi}{r} - \frac{k}{RV_0\cos\psi}\right\}^2}} \qquad (10)$$

This equation can be integrated as

$$\theta + \pi = \cos^{-1}\left[\frac{\frac{RV_0\cos\psi}{r} - \frac{k}{RV_0\cos\psi}}{\sqrt{V_0^2 - \frac{2k}{R} + \frac{k^2}{R^2V_0^2\cos^2\psi}}}\right] \qquad (11)$$

where the integration constant $-\pi$ is incorporated. (11) can be rewritten as

$$-\cos\theta = \frac{\frac{RV_0\cos\psi}{r} - \frac{k}{RV_0\cos\psi}}{\sqrt{V_0^2 - \frac{2k}{R} + \frac{k^2}{R^2V_0^2\cos^2\psi}}} \qquad (12)$$

Now at the surface of the non-rotating earth, the gravitational constant is g, therefore

$$g = \frac{k}{R^2} = 32.2 \text{ ft/sec}^2. \qquad (13)$$

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Therefore (12) becomes

$$-\cos\theta = \frac{\dfrac{R v_0 \cos\phi}{r} - \dfrac{gR}{v_0 \cos\phi}}{\sqrt{v_0^2 - 2gR + \dfrac{g^2R^2}{v_0^2\cos^2\phi}}}$$

Or

$$r = \frac{g\left(\dfrac{v_0}{g}\right)^2 \cos^2\phi}{1 - \sqrt{\left(\dfrac{v_0}{gR}\right)^2 \cos^2\phi\,(v_0^2 - 2gR) + 1}\;\cos\theta} \qquad (13)$$

This equation shows that r is a maximum when $\theta = 0$ as specified by Fig. 1, therefore our choice of the integration constant in (11) is correct. Let

$$\varepsilon^2 = \left(\frac{v_0}{gR}\right)^2 \cos^2\phi\,(v_0^2 - 2gR) + 1 \qquad (14)$$

Then (13) is

$$r = \frac{g\left(\dfrac{v_0}{g}\right)^2 \cos^2\phi}{1 - \varepsilon\cos\theta} \qquad (15)$$

But $r = R$ when $\theta = \theta_0$, therefore

$$R = \frac{g\left(\dfrac{v_0}{g}\right)^2 \cos^2\phi}{1 - \varepsilon\cos\theta_0} \qquad (16)$$

By combining (15) and (16), one has

$$r = \frac{R\,(1 - \varepsilon\cos\theta_0)}{1 - \varepsilon\cos\theta} \qquad (17)$$

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(13), (15) and (17) show that the coasting trajectory is an ellipse of $\varepsilon < 1$. This is the trajectory for missiles that will return to the earth, and is the trajectory which is considered here. (17) is specialized for the group of trajectories all having the same range determined by θ_0, then the parameter is ε, the eccentricity. We wish to calculate the trajectory among this group corresponding to the minimum energy or minimum V_0. This will be done presently:

Eliminating $\cos^2\psi$ from (14) and (16),

$$1-\varepsilon^2 = \left(\frac{V_0}{g}\right)^2 \cos^2\psi \quad \frac{2gR - V_0^2}{R^2}$$

$$= (1-\varepsilon\cos\theta_0)\left(2 - \frac{V_0^2}{gR}\right)$$

Solving for V_0^2, we have

$$V_0^2 = gR\left\{2 - \frac{1-\varepsilon^2}{1-\varepsilon\cos\theta_0}\right\} \tag{18}$$

With fixed value of θ_0, we have to determine the value of ε corresponding to the minimum of V_0. Thus this particular value ε^* of ε, must satisfy the following equation:

$$\left(\frac{\partial V_0^2}{\partial \varepsilon}\right)_{\varepsilon=\varepsilon^*} = 0.$$

Or $$-2\varepsilon^*(1-\varepsilon^*\cos\theta_0) + (1-\varepsilon^{*2})\cos\theta_0 = 0$$

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Thus the equation for ε^* is

$$\cos\theta_0\ \varepsilon^{*2} - 2\varepsilon^* + \cos\theta_0 = 0 \qquad (19)$$

The appropriate solution of this quadratic equation is

$$\varepsilon^* = \frac{1-\sin\theta_0}{\cos\theta_0} \qquad (20)$$

This is the value for the eccentricity giving the minimum initial velocity v_0^* for a given range. By substituting ε^* from (20) into (18), we have

$$v_0^{*2} = 2gR\left[1 - \frac{1-\sin\theta_0}{\cos^2\theta_0}\right] \qquad (21)$$

(14) can then be solved for the optimum value of ψ^*. The result is

$$\cos^2\psi^* = \frac{\cos^2\theta_0}{2(1-\sin\theta_0)}$$

Or

$$\tan\psi^* = \frac{1-\sin\theta_0}{\cos\theta_0} = \varepsilon^* \qquad (22)$$

The altitude H of the summit is given by (17) as

$$H = r_{max} - R = R\left[\frac{1-\varepsilon\cos\theta_0}{1-\varepsilon} - 1\right]$$

Or

$$H = R\varepsilon\,\frac{1-\cos\theta_0}{1-\varepsilon} \qquad (23)$$

Therefore for the optimum trajectory,

$$H^* = R\,\frac{1-\sin\theta_0}{\cot\frac{\theta_0}{2} - 1} \qquad (24)$$

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(19), (21), (22), and (24) are the equations to determine optimum coasting trajectory without lift.

The following table is the result of computation. The relation between v_0 and S_a is plotted in Fig. 5, curve a.

Table I

Coasting Flight without Lift

Range, S_a miles	Initial Velocity v_0 ft./sec.	Inclination ψ °	Altitude of Summit miles
100	4,099	44.64	24.85
300	7,012	43.91	73.56
500	8,942	43.19	121.20
1000	12,276	41.38	233.69
2000	16,411	37.76	431.84
3000	19,071	34.14	591.4
4000	20,959	30.52	710.4
5000	22,360	26.90	786.5
6000	23,422	23.28	818.4

b) <u>Coasting flight around the earth with lift</u>

The flight is considered to be carried out at a constant radius R. (Fig. 2) If g is the gravitational constant, u the velocity of flight, the lift produced by the wing must counterbalance the resultant of the gravitational attraction and the centrifugal force

Range miles	θ_0 °	θ_0 °	$\sin\theta_0$	$\cos\theta_0$	$\sqrt{\sin\theta_0}$	$\frac{1-\sin\theta_0}{\cos\theta_0}$	$\frac{1}{\sqrt{2}}\left(\frac{\pi}{2}-\sin^{-1}\frac{1-\sin\theta_0}{\cos\theta_0}\right)$
500	3.62	3°37'	0.06308	0.99801	0.251	0.939	0.2488
1000	7.25	7°15'	0.12620	0.99200	0.355	0.881	0.349
2000	14.49	14°29'	0.25010	0.96822	0.5005	0.775	0.484
3000	21.72	21°43'	0.37002	0.92902	0.6085	0.678	0.584
4000	28.98	28°59'	0.48456	0.87476	0.696	0.588	0.667
5000	36.22	36°13'	0.59084	0.80679	0.7685	0.507	0.735
6000	43.50	43°30'	0.68835	0.72537	0.829	0.430	0.796

Range	$\frac{\cos\theta_0}{\sqrt{1-\sin\theta_0}}$	[ ]	$t^{(1)}$ hr.
500	1.031	0.4922	0.1168
1000	1.061	0.6832	0.1715
2000	1.118	0.929	0.259
3000	1.171	1.098	0.336
4000	1.220	1.223	0.407
5000	1.262	1.318	0.469
6000	1.299	1.390	0.523

Table of Contents

2.6

Performance of Rocket Projectile

远程火箭的飞行特性

这是作者在20世纪40年代中后期分析长程火箭的飞行特性的一份资料的部分手稿。

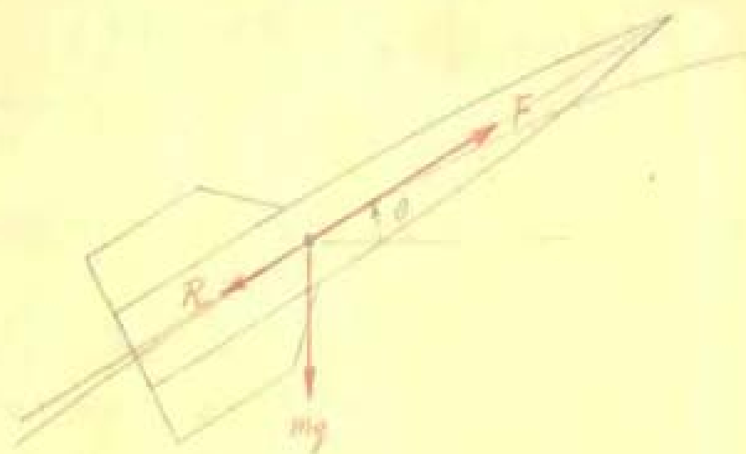
1944年，加州理工学院（Caltech）受美国陆军军械部（Army Ordnance）的委托，研究远程火箭。为此，加州理工学院喷气推进实验室（JPL）重新组织了力量，分成弹道、材料、推进和结构等四个部分，由作者负责推进方面的工作。作者在那一段时期，为远程火箭进行了多方面的分析，从发动机、火箭整体结构直到飞行轨道等，设想了各种方案，并且进行了优化分析。

在这份资料中作者对远程火箭的飞行特性，包括射程、动力飞行阶段、自由飞行阶段以及有翼滑翔阶段等进行了分析计算，并且提供了LRRP－I和LRRP－II两个实际算例。这里选印了手稿中的11页，其中9页是关于轨道、升力系数 C_l 和阻力系数 C_d 的推导和计算，2页是实例的计算。

这是按合同为美国陆军航空兵提供的一份内部报告。

Performance of Rocket Projectile

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F = thrust, lb.

R = Drag, lb. $= \frac{(\rho/g)}{2} v^2 C_D(v, y) A$

m = mass, slugs $= m_0 - \int_0^t \delta F \, dt$

m_0 = initial mass

δ = propellant consumption in slugs per lb. thrust per sec.

t = time

v = flight velocity, ft/sec.

θ = inclination of the trajectory from horizon, positive upward

g = gravitational constant, 32.2 ft/sec²

The equations of motion

$$m \frac{dv}{dt} = F - R - mg \sin\theta \qquad (1)$$

$$v \frac{d\theta}{dt} = -g \cos\theta \qquad (2)$$

Divide through eq. (1) by m_0,

$$\frac{m}{m_0}\frac{dv}{dt} = \frac{F}{m_0} - \frac{R}{m_0} - \frac{m}{m_0} g \sin\theta \qquad (3)$$

Put

$$\frac{m}{m_0} = \zeta = 1 - \int_0^t \frac{1}{c}\frac{F}{m_0}\,dt \qquad (4)$$

Then

$$\zeta \frac{dv}{dt} = \frac{F}{m_0} - \frac{R}{m_0} - \zeta g \sin\theta \qquad (5)$$

From the calculation for sounding rocket, it seems reasonable to assume,

$\frac{dv}{dt}$ = constant = a Then

$$v = v_0 + at \qquad (6)$$

The second equation (2) is

$$(v_0 + at)\frac{d\theta}{dt} = -g\cos\theta$$

or

$$-\frac{d\theta}{\cos\theta} = g\,\frac{dt}{v_0 + at}$$

If at $t=0$, $\theta = \theta_0$, then

$$\int_{\theta_0}^{\theta} \frac{d\theta}{\cos\theta} = -g\int_0^t \frac{dt}{v_0 + at}$$

or

$$\frac{1}{2}\log\frac{1+\sin\theta}{1-\sin\theta}\,\frac{1-\sin\theta_0}{1+\sin\theta_0} = -\frac{g}{a}\log\frac{v_0 + at}{v_0}$$

Then

$$\boxed{\frac{1+\sin\theta}{1-\sin\theta}\,\frac{1-\sin\theta_0}{1+\sin\theta_0} = \left(1 + \frac{at}{v_0}\right)^{-\frac{2g}{a}}} \qquad (7)$$

Put

$$\frac{1+\sin\theta_0}{1-\sin\theta_0} = k^2$$

$$1+\frac{at}{v_0} = \xi = \text{ratio of velocity at time } t \text{ to initial velocity}$$

$$\frac{a}{g} = n$$

Then (7) becomes $\quad k^2\frac{1-\sin\theta}{1+\sin\theta} = \xi^{\frac{2}{n}}$

or $\quad k^2(1-\sin\theta) = \xi^{\frac{2}{n}}(1+\sin\theta)$

$$\sin\theta(\xi^{\frac{2}{n}}+k^2) = k^2-\xi^{\frac{2}{n}}$$

$$\sin\theta = \frac{k^2-\xi^{\frac{2}{n}}}{k^2+\xi^{\frac{2}{n}}}$$

$$\cos\theta = \frac{2k\xi^{\frac{1}{n}}}{k^2+\xi^{\frac{2}{n}}} \qquad (8)$$

Then $\quad v_y = v\sin\theta = v_0\xi\sin\theta = \frac{dy}{dt}$

$$y = v_0\int_0^t \xi\sin\theta\, dt \quad , \quad \text{but } \frac{a}{v_0}dt = d\xi, \quad dt = \frac{v_0}{a}d\xi$$

$$y = \frac{v_0^2}{a}\int_1^{\xi} \xi\sin\theta\, d\xi$$

$$\frac{y}{\left(\frac{v_0^2}{a}\right)} = \int_1^{\xi} \xi\,\frac{k^2-\xi^{\frac{2}{n}}}{k^2+\xi^{\frac{2}{n}}}\,d\xi = -\frac{1}{2}(\xi^2-1)+2\int_1^{\xi}\frac{k^2\xi}{k^2+\xi^{\frac{2}{n}}}\,d\xi = \frac{y}{\left(\frac{v_0^2}{a}\right)} \qquad (9)$$

Similarly $\quad v_x = v\cos\Theta = v_0\xi\cos\Theta = \frac{dx}{dt}$

$$\frac{x}{\left(\frac{v_0^2}{a}\right)} = \int_1^{\xi} \frac{2k\xi^{\frac{1}{n}+1}}{k^2+\xi^{\frac{2}{n}}}\,d\xi \qquad (10)$$

For the case $n = 1$

$$\frac{x}{\left(\frac{v_0^2}{a}\right)} = \int_1^{\xi} \frac{2k\xi^2}{k^2+\xi^2}\,d\xi = 2k\int_1^{\xi}\left(1 - \frac{k^2}{k^2+\xi^2}\right)d\xi$$

$$\frac{x}{\left(\frac{v_0^2}{a}\right)} = 2k\left\{(\xi-1) - k\left(\tan^{-1}\frac{\xi}{k} - \tan^{-1}\frac{1}{k}\right)\right\} \qquad (11)$$

If we let $\varphi_0 = \frac{\pi}{2} - \theta_0$, $\quad$ then $k^2 = \frac{1+\cos\varphi_0}{1-\cos\varphi_0}$

$$\frac{1}{k} = \tan\frac{\varphi_0}{2}, \qquad \tan^{-1}\frac{1}{k} = \frac{\pi}{4} - \frac{\theta_0}{2} \qquad k = \cot\left(\frac{\pi}{4} - \frac{\theta_0}{2}\right)$$

$$\frac{y}{\left(\frac{v_0^2}{a}\right)} = k^2\log_e\frac{k^2+\xi^2}{k^2+1} - \frac{1}{2}(\xi^2-1) \qquad (12)$$

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Dividing (5) by g

$$n \cdot \zeta = \frac{F}{m_0 g} - \frac{R}{m_0 g} - \zeta \sin\Theta$$

Let

$$\boxed{\frac{F}{m_0 g} = f, \qquad \frac{R}{m_0 g} = r}$$

$$\zeta = 1 - \int_1^t \sigma \frac{F}{m_0}\, dt = 1 - \frac{v_0 \sigma}{n} \int_1^\xi \frac{F}{m_0} d\xi$$

$$\boxed{\zeta = 1 - \frac{v_0 \sigma}{n} \int_1^\xi f\, d\xi}$$

σ has a dimension $\frac{T}{L}$, so $v_0 \sigma$ is dimensionless

Then

$$n\left(1 - \frac{v_0 \sigma}{n} \int_1^\xi f\, d\xi\right) = f - r - \sin\Theta \left(1 - \frac{v_0 \sigma}{n} \int_1^\xi f\, d\xi\right)$$

$$n\left(1 + \frac{\sin\Theta}{n}\right) - f + r = v_0 \sigma \left(1 + \frac{\sin\Theta}{n}\right) \int_1^\xi f\, d\xi$$

$$n - \frac{f - r}{\left(1 + \frac{\sin\Theta}{n}\right)} = v_0 \sigma \int_1^\xi f\, d\xi$$

Differentiating with respect to ξ

$$\boxed{-\left(\frac{df}{d\xi} - \frac{dr}{d\xi}\right) + (f - r) \frac{d}{d\xi} \log \left(1 + \frac{\sin\Theta}{n}\right) = v_0 \sigma \left(1 + \frac{\sin\Theta}{n}\right) f}$$

$$-\frac{d(f-r)}{d\xi} + (f - r) \frac{d}{d\xi} \log \left(1 + \frac{\sin\Theta}{n}\right) = v_0 \sigma \left(1 + \frac{\sin\Theta}{n}\right)(f - r) + v_0 \sigma \left(1 + \frac{\sin\Theta}{n}\right) r$$

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Therefore

$$\frac{d(f-r)}{d\xi} + \left\{ v_0\sigma\left(1+\frac{\sin\theta}{n}\right) - \frac{d}{d\xi}\log_e\left(1+\frac{\sin\theta}{n}\right) \right\}(f-r) = -v_0\sigma\left(1+\frac{\sin\theta}{n}\right) r$$

Take the case $n=1$

$$\frac{d(f-r)}{d\xi} + \underbrace{\left\{ v_0\sigma(1+\sin\theta) - \frac{d}{d\xi}\log_e(1+\sin\theta) \right\}}_{P}(f-r) = \underbrace{-v_0\sigma(1+\sin\theta)r}_{Q}$$

$$\int P\,d\xi = v_0\sigma\int(1+\sin\theta)\,d\xi - \log_e(1+\sin\theta) \quad , \quad (1+\sin\theta) = \frac{2k^2}{k^2+\xi^2}$$

$$= v_0\sigma\cdot 2k\tan^{-1}\frac{\xi}{k} - \log_e(1+\sin\theta)$$

$$= 2kv_0\sigma\tan^{-1}\frac{\xi}{k} - \log_e(1+\sin\theta)$$

$$e^{\int P\,d\xi} = \frac{e^{2kv_0\sigma\tan^{-1}\frac{\xi}{k}}}{1+\sin\theta}$$

$$f-r = (1+\sin\theta)\,e^{-2kv_0\sigma\tan^{-1}\frac{\xi}{k}}\left\{ C - v_0\sigma\int r\,e^{2kv_0\sigma\tan^{-1}\frac{\xi}{k}}\,d\xi \right\}$$

$$f_0 - r_0 = (1+\sin\theta_0)\,e^{-2kv_0\sigma\tan^{-1}\frac{\xi}{k}}\,C$$

But $\quad f_0 - r_0 = 1+\sin\theta_0$

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Therefore

$$f-\tau=(1+k\sin\theta)\,e^{-2k\nu_0\tau\tan^{-1}\frac{\xi}{k}}\left\{e^{2k\nu_0\tau\tan^{-1}\frac{1}{k}}-\nu_0\tau\int_1^{\xi}\tau\,e^{2k\nu_0\tau\tan^{-1}\frac{\xi}{k}}\,d\xi\right\}$$

$$\zeta=e^{-2k\nu_0\tau\tan^{-1}\frac{\xi}{k}}\left\{e^{2k\nu_0\tau\tan^{-1}\frac{1}{k}}-\nu_0\tau\int_1^{\xi}\tau\,e^{2k\nu_0\tau\tan^{-1}\frac{\xi}{k}}\,d\xi\right\}$$

2

ξ is really $\frac{f}{c}$ where c = effective exhaust velocity.

So $v_0 \xi = \frac{v_0}{c}$

$$r = \frac{R}{m_0 g} = \frac{\rho(y)}{2} v^2 C_D\left(\frac{v}{s}\right) \frac{A}{m_0 g}, \quad s = \text{velocity of sound}$$

$$= \frac{\rho_0}{2} v_0^2 \frac{A}{m_0 g} \cdot \frac{\rho}{\rho_0}(y)\, \xi^2 C_D\left(\frac{v}{s}\right)$$

Let us start by assuming $v_0 = 160$ ft/sec

$c = 6400$ ft/sec.

$$v_0 \xi = \frac{1}{40}$$

Take a projectile of 10,000 lb and length-diameter ratio 10.

$$\text{Volume} = \frac{\pi}{4} d^2 \times 8d \text{ — for ogive} = 2\pi d^3$$

Assume average sp. gr. = 1.6. Then

$$2\pi d^3 \times 1.6 \times 62.3 = 10{,}000$$

$$d^3 = \frac{10{,}000}{62.3 \times 1.6 \times 2\pi} = 15.96$$

$d = 2.530$	$L = 25.30$ ft long

$$\frac{\rho_0}{2} v_0^2 \frac{A}{m_0 g} = 0.001189 \times 160^2 \times \frac{\frac{\pi}{4} \times 2.530^2}{10{,}000} = \frac{2.56 \times 11.89 \times \frac{\pi}{4} \times 2.530^2}{10{,}000}$$

$$\boxed{\frac{\rho_0}{2} v_0^2 \frac{A}{m_0 g} = 0.01516}$$

First of all, we have to change the ballisticians' unit

$$D = K_D \rho d^2 v^2 = C_D \frac{\rho}{2} \frac{\pi}{4} d^2 v^2$$

$$\therefore \quad K_D = \frac{\pi}{8} C_D \qquad \boxed{C_D = \frac{8}{\pi} K_D}$$

Take $L = 21.74$ ft., $R_e = 160 \times 21.74 \times 6380 = 1.60 \times 2.174 \times 0.6380 \times 10^7$

$$= 2.220 \times 10^7$$

$$C_f = \frac{0.455}{(7.346)^{2.58}} = 0.002652$$

Area $\pi \times 2.174 \times 5 \times 2.174$

Base area $\frac{\pi}{4} \times 2.174^2$

} Ratio 20

C_D due to skin friction = 0.05304

Assume the fins give the same amount, ΔC_D = 0.10608

M	K_D	C_D	$C_D + \Delta C_D$
0	0.090	0.2292	0.3353
0.25	0.0925	0.2324	0.3385
0.5	0.0775	0.1973	0.3034
0.75	0.064	0.1630	0.2691
0.95	0.10	0.2545	0.3606
1.0	0.1575	0.401	0.5071
1.1	0.161	0.410	0.5161
1.5	0.14	0.3566	0.4627
2.0	0.115	0.2930	0.3991
2.5	0.102	0.2600	0.3661
3.0	0.092	0.234	0.3401
3.5	0.082	0.209	0.3151
4.0	0.074	0.188	0.2941

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Gabit LRRP-I

$$Re = 160 \times 7 \times 6320 = 1.60 \times 0.7 \times 6.320 \times 10^6 = 7.15 \times 10^6$$

$$\log Re = 6.854, \qquad C_f = \frac{0.455}{(6.854)^{2.58}} = 0.00318$$

$$\text{Area ratio} = \frac{\pi \times 10.5 \times 55.5}{\frac{\pi}{4} \times 10.5^2} = \frac{4 \times 55.5}{10.5} = 21.1$$

$$\Delta C_D = 0.0672 \qquad 2\Delta C_D = 0.1344$$

M	C_{D_K}	C_D	
0	0.2292	0.3636	
0.25	0.2224	0.3568	
0.50	0.1973	0.3317	
0.75	0.1630	0.2974	
0.95	0.2545	0.3889	
1.00	0.401	0.5354	
1.1	0.410	0.5444	
1.5	0.3566	0.4910	
2.0	0.2930	0.4274	
2.5	0.2600	0.3944	
3.0	0.234	0.3684	
3.5	0.209	0.3434	
4.0	0.188	0.3224	

LRRP-II

Initial weight = 2000 #

Thrust = 4000 #

Propellant weight = 830 #
Empty weight = 970 #
Pay Load = 200 #
} 2000 #

Diameter = 2 ft.

Length ~ 16 ft.

$F = 4000$ lb.

$$R = 0.001189 \times 160^2 \pi \left(\frac{\rho}{\rho_0}\right)\left(\frac{v}{v_0}\right)^2 C_D$$

$$= 95.6 \left(\frac{\rho}{\rho_0}\right)\left(\frac{v}{v_0}\right)^2 C_D$$

$$\frac{1}{g}\frac{dv}{dt} = \frac{2000}{2000 - 32.7t}\left[2.000 - 0.0478\frac{\rho}{\rho_0}C_D\xi^2\right] - \sin\theta$$

$$\frac{d\xi}{dt} = 0.2012\left[\frac{1}{1 - 0.01165t}\left(2.000 - 0.0478\frac{\rho}{\rho_0}C_D\xi^2\right) - \sin\theta\right]$$

$$\frac{d^2\xi}{dt^2} = 0.2012\left[\frac{0.01165}{(1 - 0.01165)^2}\left(2.000 - 0.0478\frac{\rho}{\rho_0}C_D\xi^2\right) - \frac{0.0956}{1 - 0.01165t}\frac{\rho}{\rho_0}C_D\xi\left(\frac{d\xi}{dt}\right)\right.$$

$$\left. + 0.2012\frac{\cos^2\theta}{\xi}\right]$$

$\theta_0 = 82°$, $\sin\theta_0 = 0.99027$ $\cos\theta_0 = 0.1392$

$(\xi)_0 = 1$

$\left(\frac{d\xi}{dt}\right)_0 = 0.2012 \times 0.993 = 0.1995$

$\left(\frac{d^2\xi}{dt^2}\right)_0 = 0.2012\left[0.01165 \times 1.983 - 0.03452 \times 0.1995 + 0.0039\right]$

$= 0.00408$

$\xi = 1 + 0.1995t + 0.00204t^2$

$\xi(0) = 1$
$\xi(1) = 1.2015$
$\xi(2) = 1.4072$

2.7

Jet Turbine Calculation

喷气透平计算——推力增加器

作者在一份题为“Jet Turbine Calculation”（喷气透平计算）的手稿中，做了有关推力增加器的计算。工作时间大约在1941—1943年间。

与其他热机相比，液体火箭发动机能更有效地将热能转化为动能。然而，实际上在大气层的较低部分，火箭尚未达到足够高的速度，发动机向大气排出的热射流中仍然具有相当大的动能和热能未被利用。值得考虑将这种白白浪费的能量转化为有用能量的可能性，譬如一种可能是，利用射流来驱动涡轮叶片，为火箭的推进剂泵提供动力；另一种可能是，将射流用作推力增加器（thrust augmentor）的动力。

一个简单的推力增加器的主要结构，就是在发动机喷管的周围套上一个两端打开的粗管。为了充分利用发动机喷管所喷出的射流的能量，不让射流直接喷入大气，而是让它再流入那个直径较粗的管子，使其引射粗管内的冷空气一起运动，然后再排入大气。Theodore von Kármán（冯·卡门）让作者计算从热气流引射冷空气直到最后排气这样一个过程的工作性能。

这里选印了这份计算手稿中的一部分，共计13页。其中，前面2页是von Kármán给作者的提纲，要求作者计算压力损失Δp与被引射的质量流和热射流的质量流的比值μ之间的关系，后面10页是作者所做的方程推导和演算，最后一页是反映计算结果的两类曲线，说明随着质量流比值μ的变化，压力损失Δp与资用功和热射流的动能之比是怎样变化的规律。

作者在1945年编写《Jet Propulsion》（喷气推进）一书时，将上述有关推力增加器的内容编入该书第11章的第3节，该节的题目是“Methods of Utilizing Energy Lost in the Jet”（发动机喷出射流所损失的能量的利用）。

Dr Tsien

$$1+\frac{2U_1}{V^2}+\mu\left[\frac{2\gamma}{\gamma-1}\,\frac{p_0-\Delta p}{\rho_2 V^2}+\left(\frac{v}{V}\right)^2\right]=$$

$$=\frac{2\gamma^*}{\gamma^*-1}\left[\left(\frac{\mathcal{A}p_0}{\mathcal{F}_0}\right)\frac{1+\mu\, v/V}{1+\mu}-\left(\frac{\mathcal{A}p_0}{\mathcal{F}_0}\right)^2\frac{\Delta p/p_0}{1+\mu}\right]+$$

$$+\frac{1}{1+\mu}\left[1+\mu\frac{v}{V}-\frac{\mathcal{A}p_0}{\mathcal{F}_0}\frac{\Delta p}{p_0}\right]^2$$

U_1 = potential energy of jet exhaust per unit mass

V = jet exhaust velocity

μ = mass ratio: secondary massflow/primary mass flow

p_0 = atmospheric pressure

Δp = suction produced in throat of augmentor

ρ_2 = density of secondary air in throat of augmentor

v = air velocity in throat of augmentor

$\mathcal{A}$ = augmentor cross sectional area = n times jet exhaust area

$\mathcal{F}_0$ = primary thrust

$\gamma = c_p/c_v$ for air

$\gamma^* = c_p/c_v$ for mixture.

ρ_1 = density of jet exhaust; ρ_0 density of outside air.

Furthermore

$$\frac{v}{V}=\mu\,\frac{1}{n-1}\,\frac{\rho_1}{\rho_2}$$

and

$$\frac{\rho_2}{\rho_0}=\left(1-\frac{\Delta p}{p_0}\right)^{\frac{1}{1.3}}$$

Determine Δp versus μ.

v. Kármán

Limiting case $\mu = 0$

$$1 + \frac{2U}{V^2} = \frac{2\gamma^*}{\gamma^* - 1}\left[\frac{a p_0}{F_0} - \left(\frac{a p_0}{F}\right)^2 \frac{\Delta p}{p_0}\right] + \left[1 - \frac{a p_0}{F}\frac{\Delta p}{p_0}\right]^2$$

or

$$\left(1 + \frac{2U}{V^2}\right) = \frac{a p_0}{F_0}\frac{2\gamma^*}{\gamma^* - 1} x + x^2$$

$$x = 1 - \frac{a p_0}{F}\frac{\Delta p}{p_0}$$

with $\frac{2U}{V^2} = 1$

$\frac{a p_0}{F_0} = 3.7$

$\gamma^* = 1.2$

$a = 50\ \text{inch}^2$

$p_0 = \sout{2100\ \text{lb/ft}^2}$

$= 14.8\ \text{lb/inch}^2$

$F_0 = 200\ \text{lb}$

$\frac{a p_0}{F} = 3.7$

$$2 = 12 \times 3.7 x + x^2$$

$$x^2 + 44.5 x - 2 = 0$$

$$x \simeq \frac{2}{44.5} = .045$$

$$3.7 \frac{\Delta p}{p_0} = 1 - 0.045$$

$$\frac{\Delta p}{p_0} = \frac{0.955}{3.7} = .256$$

$p_0 = 2100\ \text{lb/ft}^2$

$\Delta p = 540\ \text{lb/ft}^2 = 104''\ H_2O$

1

Jet Turbine Calculation

p_0 = atmospheric pressure

ρ_0 = atmospheric density

T_0 = atmospheric temperature

$\gamma = c_p/c_v$ for air $= 1.4$

$p_0 - \Delta p$ = pressure in the throat of the augmentor

β = polytropic expansion ratio $= 1.3$. (taken into account losses)

Call Δp = pressure drop across the turbine wheel.

then ρ_2 = density of air in the throat of augmentor

$$\frac{\rho_2}{\rho_0} = \left(\frac{p_0 - \Delta p}{p_0}\right)^{\frac{1}{1.3}} = \left(1 - \frac{\Delta p}{p_0}\right)^{\frac{1}{1.3}}$$

v = air velocity in throat of augmentor

V = jet exhaust velocity

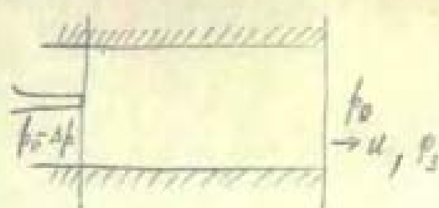
μ = secondary mass flow / primary mass flow

$$\mu = \frac{v\,(n-1)\,\rho_2}{V\ \ \rho_1}\ , \quad \text{or} \quad \frac{v}{V} = \mu \frac{1}{n-1} \frac{\rho_1}{\rho_2}$$

where ρ_1 = jet density, $n = \dfrac{\text{augmentor area}}{\text{jet exhaust area}}$

2

Then



the equation of continuity

$$(n-1)v\rho_2 + V\rho_1 = nu\rho_3$$

the equation of momentum

$$(n-1)v\rho_2 v + V\rho_1 V = nu\rho_3 u + n\Delta p$$

The energy equation

U_1 = potential energy of jet exhaust / unit mass

$$\left(U_1+\frac{1}{2}V^2\right)\rho_1 V + \left(\frac{\gamma}{\gamma-1}\frac{p_0-\Delta p}{\rho_2}+\frac{1}{2}v^2\right)(n-1)v\rho_2$$

$$=\left(\frac{\gamma_3}{\gamma_3-1}\frac{p_0}{\rho_3}+\frac{1}{2}u^2\right)\left[(n-1)v\rho_2+V\rho_1\right]$$

T_2 = temperature of air in the throat of augmenter

$$= T_0\left(1-\frac{\Delta p}{p_0}\right)^{\frac{0.3}{1.3}}$$

γ_3 = the ratio of specific heats of the gas-air mixture

$$= \frac{c_p\mu + c_p'}{\mu+1}\,\frac{\mu+1}{c_v\mu+c_v'} = \frac{\gamma\mu+\frac{c_p'}{c_v}}{\mu+\frac{c_v'}{c_v}} = \frac{\gamma\mu+\gamma'\frac{c_v'}{c_v}}{\mu+\frac{c_v'}{c_v}}$$

$$\gamma_3 = \frac{\gamma\mu + x_1}{\mu + x_2}, \qquad x_1 = \gamma' \frac{c_v'}{c_v}, \qquad x_2 = \frac{c_v'}{c_v}$$

The equation of momentum can be written as

$$(n-1)v^2\rho_2 + V^2\rho_1 = u\left[(n-1)v\rho_2 + V\rho_1\right] + n\,\Delta p$$

or

$$u = \frac{(n-1)v^2\rho_2 + V^2\rho_1 - n\Delta p}{(n-1)v\rho_2 + V\rho_1}$$

$$\frac{1}{\rho_3} = \frac{nu}{(n-1)v\rho_2 + V\rho_1} = \frac{n\left[(n-1)v^2\rho_2 + V^2\rho_1 - n\Delta p\right]}{\left[(n-1)v\rho_2 + V\rho_1\right]^2}$$

Therefore the energy equation gives

$$\frac{(U_1 + \frac{1}{2}V^2)\rho_1 V + \left(\frac{\gamma}{\gamma-1}\frac{p_0 - \Delta p}{\rho_2} + \frac{1}{2}v^2\right)(n-1)v\rho_2}{(n-1)v\rho_2 + V\rho_1}$$

$$= \frac{\gamma_3}{\gamma_3 - 1}\,\frac{p_0 n\left[(n-1)v^2\rho_2 + V^2\rho_1 - n\Delta p\right]}{\left[(n-1)v\rho_2 + V\rho_1\right]^2} + \frac{1}{2}\,\frac{\left[(n-1)v^2\rho_2 + V^2\rho_1 - n\Delta p\right]^2}{\left[(n-1)v\rho_2 + V\rho_1\right]^2}$$

4

$$\left[(n-1)v\rho_2+V\rho_1\right]\left[\left(u_1+\frac{1}{2}V^2\right)\rho_1 V+\left(\frac{\gamma}{\gamma-1}\frac{p_0-\Delta p}{\rho_2}+\frac{1}{2}v^2\right)(n-1)v\rho_2\right]$$

$$=\left[(n-1)v^2\rho_2+V^2\rho_1-n\Delta p\right]\left\{\frac{\gamma_2}{\gamma_2-1}p_0 n+\frac{1}{2}\left[(n-1)v^2\rho_2+V^2\rho_1-n\Delta p\right]\right\}$$

$$\left[(n-1)\frac{v}{V}\frac{\rho_2}{\rho_1}+1\right]\left[\left(\frac{u_1}{V^2}+\frac{1}{2}\right)+\left(\frac{\gamma}{\gamma-1}\frac{p_0-\Delta p}{\rho_2 V^2}+\frac{1}{2}\frac{v^2}{V^2}\right)(n-1)\frac{v}{V}\frac{\rho_2}{\rho_1}\right]$$

$$=\left[(n-1)\frac{v^2}{V^2}\frac{\rho_2}{\rho_1}+1-n\frac{\Delta p}{V^2\rho_1}\right]\left\{\frac{\gamma_2}{\gamma_2-1}\frac{p_0 n}{\rho_1 V^2}+\frac{1}{2}\left[(n-1)\frac{v^2}{V^2}\frac{\rho_2}{\rho_1}+1-\frac{n\Delta p}{\rho_1 V^2}\right]\right\}$$

$$(n-1)\frac{v}{V}\frac{\rho_2}{\rho_1}=(n-1)\frac{\rho_2}{\rho_1}\mu\frac{1}{n-1}\frac{\rho_1}{\rho_2}=\mu$$

$$(\mu+1)\left[\left(\frac{2u_1}{V^2}+1\right)+\left(\frac{2\gamma}{\gamma-1}\frac{p_0-\Delta p}{\rho_2 V^2}+\frac{v^2}{V^2}\right)\mu\right]$$

$$=\left[\mu\frac{v}{V}+1-n\frac{\Delta p}{V^2\rho_1}\right]\left\{\frac{2\gamma_2}{\gamma_2-1}\frac{a p_0}{q_0}+\left[\mu\frac{v}{V}+1-\frac{n\Delta p}{V^2\rho_1}\right]\right\}$$

$$1+\frac{2u_1}{V^2}+\mu\left(\frac{2\gamma}{\gamma-1}\frac{p_0-\Delta p}{\rho_2 V^2}+\frac{v^2}{V^2}\right)$$

$$=\frac{2\gamma_2}{\gamma_2-1}\frac{a p_0}{q_0}\left[\frac{1+\mu\frac{v}{V}}{1+\mu}-\frac{a p_0}{q_0}\frac{\Delta p/p_0}{1+\mu}\right]$$

$$+\frac{1}{1+\mu}\left[1+\mu\frac{v}{V}-\frac{a p_0}{q_0}\frac{\Delta p'}{p_0}\right]^2$$

$$\frac{p_0-\Delta p}{\rho_2 V^2}=\frac{n p_0}{\rho_1 V^2}\,\frac{1-\frac{\Delta p}{p_0}}{n\rho_2/\rho_1}=\frac{a p_0}{\mathfrak{F}_0}\,\frac{1-\frac{\Delta p}{p_0}}{n}\,\frac{v}{V}\,\frac{n-1}{\mu}$$

$$=\frac{a p_0}{\mathfrak{F}_0}\left(1-\frac{\Delta p}{p_0}\right)\frac{v}{V}\,\frac{n-1}{n}\,\frac{1}{\mu}$$

Our equation then becomes

$$1+\frac{2u_1}{V^2}+\frac{2\gamma}{\gamma-1}\,\frac{a p_0}{\mathfrak{F}_0}\left(1-\frac{\Delta p}{p_0}\right)\frac{v}{V}\,\frac{n-1}{n}+\mu\frac{v^2}{V^2}$$

$$=\frac{2\gamma_2}{\gamma_2-1}\,\frac{a p_0}{\mathfrak{F}_0}\left[\frac{1+\mu\frac{v}{V}}{1+\mu}-\frac{a p_0}{\mathfrak{F}_0}\,\frac{\frac{\Delta p}{p_0}}{1+\mu}\right]$$

$$+\frac{1}{1+\mu}\left[1+\mu\frac{v}{V}-\frac{a p_0}{\mathfrak{F}_0}\,\frac{\Delta p}{p_0}\right]^2$$

$$(1+\mu)\frac{1+\frac{2u_1}{V^2}+\mu\frac{v^2}{V^2}}{\left(\frac{a p_0}{\mathfrak{F}_0}\right)^2}+\frac{2\gamma}{\gamma-1}\,\frac{v}{V}\,\frac{n-1}{n}\,\frac{1}{\frac{a p_0}{\mathfrak{F}_0}}\left(1-\frac{\Delta p}{p_0}\right)(1+\mu)$$

$$=\frac{2\gamma_2}{\gamma_2-1}\left(\frac{1+\mu\frac{v}{V}}{\frac{a p_0}{\mathfrak{F}_0}}-\frac{\Delta p}{p_0}\right)+\left(\frac{1+\mu\frac{v}{V}}{\frac{a p_0}{\mathfrak{F}_0}}-\frac{\Delta p}{p_0}\right)^2$$

$$\left(\frac{\Delta p}{p_0}\right)^2 + \left[\frac{2\gamma}{\gamma-1}\frac{v}{V}\frac{n-1}{n}\frac{1+\mu}{\frac{\Delta p_0}{q_0}} - \frac{2\gamma}{\gamma-1} - 2\frac{1+\mu\frac{v}{V}}{\frac{\Delta p_0}{q_0}}\right]\frac{\Delta p}{p_0}$$

$$+ \left[\left(\frac{1+\mu\frac{v}{V}}{\frac{\Delta p_0}{q_0}}\right)^2 + \frac{2\gamma}{\gamma-1}\frac{1+\mu\frac{v}{V}}{\frac{\Delta p_0}{q_0}} - \frac{2\gamma}{\gamma-1}\frac{v}{V}\frac{n-1}{n}\frac{1+\mu}{\frac{\Delta p_0}{q_0}} - (1+\mu)\frac{1+\frac{2\gamma_1}{V^2}\gamma\mu\frac{v^2}{V^2}}{\left(\frac{\Delta p_0}{q_0}\right)^2}\right] = 0$$

The thrust $\mathfrak{F} = (\gamma-1)\mu\rho_2 v + \mathfrak{F}_0 - n\Delta p$

$$\frac{\mathfrak{F}}{\mathfrak{F}_0} = 1 + (n-1)\frac{\rho}{\rho_0}\left(\frac{v}{V}\right)^2 - \frac{n\Delta p}{V^2\rho_1}$$

$$\boxed{\frac{\mathfrak{F}}{\mathfrak{F}_0} = 1 + \mu\frac{v}{V} - \frac{\Delta p_0}{q_0}\frac{\Delta p}{p_0}}$$

$$\frac{v}{V} = \mu\frac{1}{n-1}\frac{\rho}{\rho_2} = \mu\frac{1}{n-1}\frac{\rho}{\rho_0}\frac{\rho_0}{\rho_2} = \mu\frac{1}{n-1}\frac{\rho}{\rho_0}\left(1-\frac{\Delta p}{p_0}\right)^{-\frac{1}{1.3}}$$

$$\cong \mu\frac{1}{n-1}\frac{\rho}{\rho_0}\left(1+\frac{1}{1.3}\frac{\Delta p}{p_0}\right)$$

$$\left(\frac{\Delta p}{p_0}\right)^2+\left[\frac{2\gamma}{\gamma-1}\mu\frac{1}{n-1}\frac{\rho_1}{\rho_0}\left(1+\frac{1}{1.3}\frac{\Delta p}{p_0}\right)^{\frac{n-1}{n}}\frac{1+\mu}{\frac{a p_0}{\rho_0}}-\frac{2\gamma_1}{\gamma_1-1}-2\,\frac{1+\frac{\mu^2}{n-1}\frac{\rho_1}{\rho_0}\left(1+\frac{1}{1.3}\frac{\Delta p}{p_0}\right)}{\frac{2p_0}{\rho_0}}\right]\frac{\Delta p}{p_0}$$

$$=\left(\frac{\Delta p}{p_0}\right)^2\left[1+\frac{2\gamma}{\gamma-1}\frac{\mu}{n}\frac{1+\mu}{\frac{a p_0}{\rho_0}}\frac{\rho_1}{\rho_0}\frac{1}{1.3}-2\frac{\mu^2}{n-1}\frac{\rho_1}{\rho_0}\frac{1}{\frac{a p_0}{\rho_0}}\frac{1}{1.3}\right]$$

$$+\left[\frac{2\gamma}{\gamma-1}\frac{\mu}{n}\frac{1+\mu}{\frac{a p_0}{\rho_0}}\frac{\rho_1}{\rho_0}-\frac{2\gamma_1}{\gamma_1-1}-\frac{2+\frac{\mu^2}{n-1}\frac{\rho_1}{\rho_0}}{\frac{a p_0}{\rho_0}}\right]\frac{\Delta p}{p_0}$$

$$\left(\frac{1+\mu\frac{v}{V}}{\frac{a p_0}{\rho_0}}\right)^2-(1+\mu)\frac{1+\frac{2U_1}{V^2}+\mu\frac{v^2}{V^2}}{\left(\frac{a p_0}{\rho_0}\right)^2}=\frac{1}{\left(\frac{a p_0}{\rho_0}\right)^2}\left[1+2\mu\frac{v}{V}+\mu^2\frac{v^2}{V^2}-1-\frac{2U_1}{V^2}-\mu\frac{v^2}{V^2}\right.$$

$$\left.-\mu-\mu\frac{2U_1}{V^2}-\mu^2\frac{v^2}{V^2}\right]$$

$$=\frac{1}{\left(\frac{a p_0}{\rho_0}\right)^2}\left[2\mu\frac{\mu}{n-1}\frac{\rho_1}{\rho_0}\left(1+\frac{1}{1.3}\frac{\Delta p}{p_0}\right)-\left(\mu+\mu\frac{2U_1}{V^2}+\frac{2U_1}{V^2}\right)\right.$$

$$\left.-\mu\frac{\mu^2}{(n-1)^2}\left(\frac{\rho_1}{\rho_0}\right)^2\left(1+\frac{1}{1.3}\frac{\Delta p}{p_0}\right)^2\right]$$

$$=\frac{1}{\left(\frac{a p_0}{\rho_0}\right)^2}\left[2\mu\frac{\mu}{n-1}\frac{\rho_1}{\rho_0}-\mu-\mu\frac{2U_1}{V^2}-\frac{2U_1}{V^2}-\mu\frac{\mu^2}{(n-1)^2}\left(\frac{\rho_1}{\rho_0}\right)^2\right.$$

[illegible]

The coefficient of $\left(\frac{\Delta p}{p_0}\right)^2$ is

$$1+\frac{2\gamma}{\gamma-1}\frac{\mu}{\gamma_0}\frac{1+\mu}{\frac{\Delta p_2}{p_0}}\frac{\rho}{\rho_0}\frac{1}{1.3}-2\frac{\mu^2}{\gamma_0-1}\frac{\rho}{\rho_0}\frac{1}{\frac{\Delta p_2}{p_0}}\frac{1}{1.3}-\mu\frac{\mu^2}{(\gamma_0-1)^2}\left(\frac{\rho}{\rho_0}\right)^2\frac{1}{1.3^2}\frac{1}{\left(\frac{\Delta p_2}{p_0}\right)^2}=A_1$$

The coefficient of $\left(\frac{\Delta p}{p_0}\right)$ is

$$\frac{2\gamma}{\gamma-1}\frac{\mu}{\gamma_0}\frac{1+\mu}{\frac{\Delta p_2}{p_0}}\frac{\rho}{\rho_0}-\frac{2\gamma_0}{\gamma_0-1}-\frac{2+\frac{\mu^2}{\gamma_0-1}\frac{\rho}{\rho_0}}{\frac{\Delta p_2}{p_0}}+\left[2\frac{\mu^2}{\gamma_0-1}\frac{\rho}{\rho_0}\frac{1}{1.3}-2\mu\frac{\mu^2}{(\gamma_0-1)^2}\left(\frac{\rho}{\rho_0}\right)^2\frac{1}{1.3}\right]\frac{1}{\left(\frac{\Delta p_2}{p_0}\right)^2}$$

$$+\frac{2\gamma_0}{\gamma_0-1}\frac{\mu^2}{\gamma_0-1}\frac{\rho}{\rho_0}\frac{1}{1.3}\frac{1}{\frac{\Delta p_2}{p_0}}-\frac{2\gamma}{\gamma-1}\frac{\mu}{\gamma_0}\frac{\rho}{\rho_0}\frac{1}{1.3}\frac{1+\mu}{\frac{\Delta p_2}{p_0}}=A_2$$

The coefficient of 1 is

$$\frac{1}{\left(\frac{\Delta p_2}{p_0}\right)^2}\left[2\mu\frac{\mu}{\gamma_0-1}\frac{\rho}{\rho_0}-\mu-\mu\frac{2U_1}{V^2}-\frac{2U_1}{V^2}-\mu\frac{\mu^2}{(\gamma_0-1)^2}\left(\frac{\rho}{\rho_0}\right)^2\right]+\frac{2\gamma_0}{\gamma_0-1}\frac{1}{\frac{\Delta p_2}{p_0}}\frac{\rho}{\rho_0}$$

$$+\mu^2\frac{2\gamma_0}{\gamma_0-1}\frac{1}{\gamma_0-1}\frac{\rho}{\rho_0}\frac{1}{\frac{\Delta p_2}{p_0}}-\frac{2\gamma}{\gamma-1}\frac{\mu}{\gamma_0}\frac{\rho}{\rho_0}\frac{1+\mu}{\frac{\Delta p_2}{p_0}}=A_3$$

$$A_1=(1+\mu)\left[1+\frac{2\gamma}{\gamma-1}\frac{p_0}{p_0}\frac{1}{1.3}\frac{T}{T_0}\frac{1}{\frac{Q_0}{c_pT_0}}\right]-\mu\left[1+\frac{\mu}{(\gamma-1)}\frac{1}{1.3}\frac{T}{T_0}\frac{1}{\frac{Q_0}{c_pT_0}}\right]^2$$

$$A_2=\frac{2\gamma}{\gamma-1}\frac{p_0}{p_0}\frac{1+\mu}{\frac{Q_0}{c_pT_0}}\frac{T}{T_0}\left(1-\frac{1}{1.3}\right)+\frac{2\gamma}{\gamma-1}\left\{\frac{\mu^2}{\gamma-1}\frac{T}{T_0}\frac{1}{1.3}\frac{1}{\frac{Q_0}{c_pT_0}}-1\right\}$$

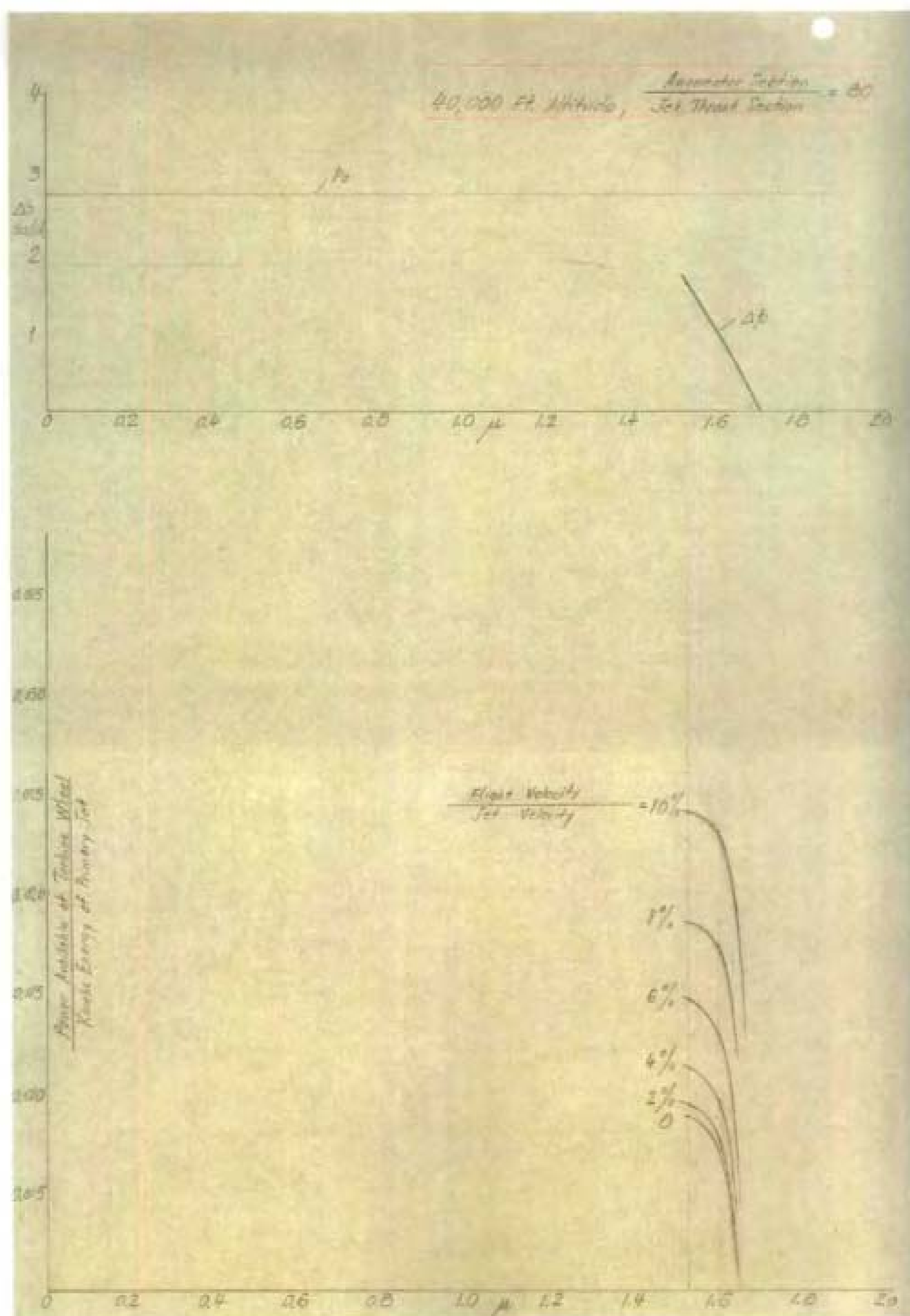
$$-\frac{2+\frac{\mu^2}{\gamma-1}\frac{T}{T_0}}{\frac{Q_0}{c_pT_0}}+2\left\{1-\frac{2\gamma}{\gamma-1}\frac{T}{T_0}\frac{1}{1.3}\right\}\frac{\mu^2}{(\gamma-1)}\frac{1}{1.3}\frac{T}{T_0}\frac{1}{\left(\frac{Q_0}{c_pT_0}\right)^2}$$

$$A_3=\frac{1}{\left(\frac{Q_0}{c_pT_0}\right)^2}\left[\frac{\mu^2}{\gamma-1}\frac{T}{T_0}\left\{2-\frac{\mu}{(\gamma-1)}\frac{T}{T_0}\right\}-\mu-(1+\mu)\frac{2\mu_0}{V^2}\right]+\frac{2\gamma}{\gamma-1}\frac{1}{\frac{Q_0}{c_pT_0}}\left[1+\frac{\mu^2}{\gamma-1}\frac{T}{T_0}\right]$$

$$-\frac{2\gamma}{\gamma-1}\frac{p_0}{p_0}\frac{T}{T_0}\frac{1+\mu}{\frac{Q_0}{c_pT_0}}$$

$$\frac{T}{T_0}=\left(1+\frac{\mu^2}{\gamma-1}\frac{T}{T_0}\right)+\left[\frac{\mu^2}{\gamma-1}\frac{T}{T_0}\frac{1}{1.3}-\frac{Q_0}{c_pT_0}\right]\frac{p_0}{p_0}$$

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40,000 Ft. Altitude, Accumulator Section / Jet Throat Section = 80
4
3
2
1
0
0.2
0.4
0.6
0.8
1.0 μ
1.2
1.4
1.6
1.8
2.0
Δp
Flight Velocity / Jet Velocity = 10%
8%
6%
4%
2%
0

工 程 控 制 论

3.1
Optimum Thrust Programming for a Sounding Rocket
探空火箭推力的优化规划

这是作者在 1951 年和他的研究生 R. C. Evans 合作发表的“Optimum Thrust Programming for a Sounding Rocket”（探空火箭推力的优化规划）一文的部分手稿。

怎样使探空火箭经济而有效地达到最大高度是一个现实问题，应该对火箭在上升过程中的推力随时间的变化做一个适宜的规划。

G. Hamel 早在 1927 年就研究过这一问题，但 Hamel 的论文写得太简单而不容易看懂。于是，作者提出了下面一个更一般的问题，即：在给定推进剂性能的条件下，若假设火箭的排气速度是常数，要把指定质量的设备送上指定的高度，那么应该采取什么样的推力随时间变化的关系，才能让最初的质量（包括火箭加燃料）达到最小值。

作者曾经做过多种试探而均不满意，手稿多达一百多页。这里只选印了其中的 12 页。前面的 6 页只反映作者推演到第 110 页以后的两种试探，然而作者分别写上了“Unsatisfactory !!!”（不满意！！！）和“(N. G.)”的结论，这里“N. G.”是“行不通”的意思。后面的 6 页上的提法则是作者比较满意的方案，后来发表的论文便是从这里出发，给出了通解，又考虑了空气阻力随速度的平方成正比和随速度的一次方成正比的两种情况，分别给出了数值算例。

Variational Problem

Formulation of the Problem (I) Unsatisfactory !!! We try to determine the best way of variation of thrust [assuming here that the exhaust velocity is constant] so that by starting out with zero velocity to reach the altitude "h" with the final mass M_e, with minimum starting mass M_a.

Ref: G. Hamel: „Über eine mit dem Problem der Rakete zusammenhängende Aufgabe der Variationsrechnung." ZAMM 7: 451-452, (1927)

Formation of Variation Problem: Let us put M = instantaneous mass of the rocket, C = exhaust velocity of the rocket, $u = \frac{ds}{dt}$ = velocity of the rocket, $W(s,u)$ = drag of the rocket. Then the differential equation of motion is

$$M\frac{du}{dt} + C\frac{dM}{dt} + W(s,u) + Mg = 0. \qquad (1)$$

Now this equation can be taken as a linear differential equation in M and put into the form

$$\frac{dM}{dt} + \left(\frac{1}{C}\frac{du}{dt} + \frac{g}{C}\right)M = -\frac{1}{C}W(s,u) \qquad (2)$$

This can be integrated as

$$M = e^{-\left(\frac{u}{c} + \frac{gt}{c}\right)} \int_0^t -\frac{1}{c} W(s,u)\, e^{\frac{u}{c} + \frac{gt}{c}}\, dt$$

$$+ \text{Constant} \cdot e^{-\left(\frac{u}{c} + \frac{gt}{c}\right)} \cdot$$

$$M_a = \text{Constant} \cdot e^{-\frac{u_a}{c}} \qquad (3)$$

$$M_e = e^{-\left(\frac{u_e}{c} + \frac{gt_e}{c}\right)} \int_0^{t_e} -\frac{1}{c} W(s,u)\, e^{\frac{u}{c} + \frac{gt}{c}}\, dt$$

$$+ \text{constant} \cdot e^{-\left(\frac{u_e}{c} + \frac{gt_e}{c}\right)} \qquad (4)$$

$$\therefore (4), \quad \text{Constant} = M_e\, e^{\frac{u_e}{c} + \frac{gt_e}{c}} + \int_0^{t_e} \frac{1}{c} W(s,u)\, e^{\frac{u}{c} + \frac{gt}{c}}\, dt$$

$$\therefore \quad M_a = e^{-\frac{u_a}{c}} \int_0^{t_e} \frac{1}{c} W(s,u)\, e^{\frac{u}{c} + \frac{gt}{c}}\, dt + M_e\, e^{-\frac{u_a}{c} + \frac{u_e}{c} + \frac{gt_e}{c}} \qquad (5)$$

Now we have assumed that $u_a = 0$,

$$\therefore \quad M_a = \int_0^{t_e} \frac{1}{c} W(s,u)\, e^{\frac{u}{c} + \frac{gt}{c}}\, dt + M_e\, e^{\frac{u_e}{c} + \frac{gt_e}{c}} \qquad (6)$$

This can be put into the form

$$M_a = \int_0^{t_e} f\left(s, \frac{ds}{dt}, t\right) dt + F\left[\left(\frac{ds}{dt}\right)_e, t_e\right] \qquad (7)$$

Now during the coasting flight, (1) becomes

$$M_e \frac{du}{dt} + W(s,u) + M_e g = 0$$

or $$M_e \frac{du}{ds} u + W(s,u) + M_e g = 0 \qquad (8)$$

Hence with the condition that $u=0$ for $s=h$, we have the integral of (8)

$$u_e = \Psi(s_e) \qquad (9)$$

In fact, if "h" is high, we the the following approximate relation

$$u_e = \sqrt{2g(h-s_e)} \qquad (9a)$$

We can now state the variation problem as follows: To minimize M_a, & at the same time the end point must lie on the curve (9). [or (9a)]

Solution of the Variational Problem: We have from (7)

$$\delta M_a = \left[\frac{\partial f}{\partial u}\delta s\right]_{t_e} + \int_0^{t_e}\left[\frac{\partial f}{\partial s} - \frac{d}{dt}\left(\frac{\partial f}{\partial u}\right)\right]\delta s\, dt$$

$$+ \frac{1}{c} M_e\, e^{\frac{u_e}{c} + \frac{g t_e}{c}} \left(\frac{du_e}{ds_e} + g\frac{dt_e}{ds_e}\right)\delta s$$

$$= \left[\frac{\partial f}{\partial u} + \frac{1}{c} M_e\, e^{\frac{u_e}{c} + \frac{g t_e}{c}} \left(\frac{du_e}{ds_e} + g\frac{1}{\frac{du_e}{dt_e}}\right)\right]\delta s$$

$$+ \int_0^{t_e}\left[\frac{\partial f}{\partial s} - \frac{d}{dt}\left(\frac{\partial f}{\partial u}\right)\right]\delta s\, dt.$$

Therefore for $\delta M_e = 0$, we have to make

$$\frac{\partial f}{\partial s} - \frac{d}{dt}\left(\frac{\partial f}{\partial u}\right) = 0 \qquad (10a)$$

and

$$\left(\frac{\partial f}{\partial u}\right)_{t=t_e} + \frac{1}{c} M_e e^{\frac{u_e}{c} + \frac{g t_e}{c}} \left(\frac{du_e}{ds_e} + g\frac{1}{u_e}\right) = 0 \qquad (11a)$$

But

$$\frac{\partial f}{\partial u} = \frac{1}{c}\left[\frac{\partial W}{\partial u} + \frac{W(s,u)}{c}\right] e^{\frac{u}{c} + \frac{gt}{c}}$$

$\therefore$ (11a) can be written as

$$\left(\frac{\partial W}{\partial u}\right)_{s=s_e} + \frac{W(s_e, u_e)}{c} + M_e\left(\frac{du_e}{ds_e} + \frac{g}{u_e}\right) = 0 \qquad (11)$$

where s_e & u_e are connected by the relation (9).

Now (10a) can be re-written as

X

$$\frac{\partial W}{\partial s} - \left[\frac{d}{dt}\left(\frac{\partial W}{\partial u}\right) + \frac{1}{c}\frac{d}{dt}W + \frac{g}{c}\left(\frac{\partial W}{\partial u} + \frac{1}{c}W\right)\right] = 0. \qquad (10)$$

But

$$W = \frac{1}{2}\rho\,\sigma(s)R(u) = k\,\sigma(s)R(u)$$

$$\frac{\partial W}{\partial s} = k\,\sigma'(s)R(u), \qquad \frac{\partial W}{\partial u} = k\,\sigma(s)R'(u)$$

$$\frac{dW}{dt} = k\left[\sigma'(s)\,u\,R(u) + \sigma(s)R'(u)\frac{du}{dt}\right]$$

$$\frac{d}{dt}\left(\frac{\partial W}{\partial u}\right) = k\left[\sigma'(s)\,u\,R'(u) + \sigma(s)R''(u)\frac{du}{dt}\right]$$

Therefore

$$\sigma'(s)n(u) - \Big[\sigma'(s)u n'(u) + \sigma(s)n''(u)\frac{du}{dt} + \frac{1}{c}\sigma'(s)u\,n(u)$$

$$+ \frac{1}{c}\,\sigma(s)n'(u)\frac{du}{dt} + \frac{g}{c}\Big\{\sigma(s)n'(u) + \frac{\sigma(s)n(u)}{c}\Big\}\Big] = 0.$$

or

$$\left(1-\frac{u}{c}\right)\sigma'(s)n(u) - \frac{1}{c}\left(\frac{du}{dt}+g\right)\sigma(s)n(u)$$

$$- \sigma'(s)u n'(u) - \frac{g}{c^2}\,\sigma(s)n'(u) - \sigma(s)n''(u)\frac{du}{dt} = 0 \qquad (10)$$

Unsuccessful !!

(II)

The problem can be formulated in the following way: To reach the altitude s_e with a velocity u_e and final mass M_e, it is required to determine the way of using the thrust, so that the initial mass is a minimum.

The equation of motion can be written as

$$M\frac{du}{ds}\cdot u + C\frac{dM}{ds}u + W(s,u) + Mg = 0 \qquad (1)$$

or

$$\frac{dM}{ds} + \left(\frac{1}{c}\frac{du}{ds} + \frac{g}{cu}\right)M = -\frac{W(s,u)}{cu}$$

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Therefore

$$M = e^{-\left\{\frac{u}{c} + \frac{g}{c}\int_0^s \frac{ds}{u}\right\}} \int_0^s -\frac{W}{cu} e^{\left(\frac{u}{c} + \frac{g}{c}\int_0^s \frac{ds}{u}\right)} ds$$

$$+ \; k \, e^{-\left\{\frac{u}{c} + \frac{g}{c}\int_0^s \frac{ds}{u}\right\}}$$

$$M_e = e^{-\left\{\frac{u_e}{c} + \frac{g}{c}\int_0^{s_e} \frac{ds}{u}\right\}} \int_0^{s_e} -\frac{W}{cu} e^{\left(\frac{u}{c} + \frac{g}{c}\int_0^s \frac{ds}{u}\right)} ds$$

$$+ \; k \, e^{-\left\{\frac{u_e}{c} + \frac{g}{c}\int_0^{s_e} \frac{ds}{u}\right\}}$$

$$M_a = k$$

$$\boxed{M_a = \int_0^{s_e} \frac{W}{cu} e^{\left(\frac{u}{c} + \frac{g}{c}\int_0^s \frac{ds}{u}\right)} ds + M_e \, e^{\frac{u_e}{c} + \frac{g}{c}\int_0^{s_e} \frac{ds}{u}}}$$

In this form, we can easily use the Ritz method to determine the best function $u(s)$ and so from (1), $M(s)$ or $M(t)$.

Variational Problem of Sounding Rocket

M = instantaneous mass at t

c = constant exhaust velocity of rocket motor

ξ = height at t

$\dot{\xi} = d\xi/dt$ = velocity at t.

$W(\xi, \dot{\xi})$ = drag of rocket at height ξ and velocity $\dot{\xi}$

Then the equation of motion gives

$$\frac{dM}{dt} + \frac{1}{c}\left(\frac{d\dot{\xi}}{dt} + g\right)M = -\frac{W(\xi,\dot{\xi})}{c} \qquad (1)$$

The initial and final conditions are

$$\left.\begin{array}{llll} t=0, & M=M', & \xi=0, & \dot{\xi}=\dot{\xi}_0 \\ t=t_1, & M=M_1, & \xi=\xi_1, & \dot{\xi}=\dot{\xi}_1 \end{array}\right\} \qquad (2)$$

(1) and (2) gives

$$M' = e^{-\frac{\dot{\xi}_0}{c}}\left\{\int_0^{t_1}\frac{W(\xi,\dot{\xi})}{c}\, e^{\frac{1}{c}(\dot{\xi}+gt)}\,dt + M_1 e^{\frac{1}{c}(\dot{\xi}_1+gt_1)}\right\} \qquad (3)$$

If M_0 is the initial weight including boosting, then

$$M_0 = M' e^{\frac{\dot{\xi}_0}{c}} \qquad (4)$$

Then (3) and (4) give

$$\boxed{M_0 = \int_0^{t_1}\frac{W(\xi,\dot{\xi})}{c}\, e^{\frac{1}{c}(\dot{\xi}+gt)}\,dt + M_1 e^{\frac{1}{c}(\dot{\xi}_1+gt_1)}} \qquad (5)$$

The Problem: To find condition on $\xi(t)$ such that the rocket will reach a given height with given M_1, c, g and W-function at minimum M_0. To reach the given height,

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$$\dot{s}_1 = \phi(s_1) \qquad (6)$$

where ϕ is specified.

Now let $s = s(t)$ be the required function so that $s(0) = 0$.

Let $\eta = \eta(t)$, $\eta(0) = 0$ (7)

Construct the "neighboring" functions of $s(t)$ as

$$\bar{s}(t) = s(t) + k(\varepsilon)\eta(t) \qquad (8)$$

where k is a parameter but not a function of t. The burning time for the "neighboring" rocket design is $(t_1+\varepsilon)$. Thus

$k(0)=0$, and $k(\varepsilon) \cong k'(0)\varepsilon$

$$\bar{s}_1 = \bar{s}(t_1+\varepsilon) = s(t_1) + \varepsilon\,\dot{s}(t_1) + k'(0)\varepsilon\,\eta(t_1) \qquad (9)$$

$$\dot{\bar{s}}_1 = \dot{\bar{s}}(t_1+\varepsilon) = \dot{s}(t_1) + \varepsilon\,\ddot{s}(t_1) + k'(0)\varepsilon\,\dot{\eta}(t_1) \qquad (10)$$

$\bar{s}_1$ and $\dot{\bar{s}}_1$ must satisfy (6). Therefore by taking only first order quantities,

$$\dot{\bar{s}}_1 = \dot{s}(t_1) + \varepsilon\,\ddot{s}(t_1) + k'(0)\varepsilon\,\dot{\eta}(t_1)$$

$$= \dot{s}(t_1) + \left(\frac{d\phi}{ds_1}\right)_{s_1}\left[s(t_1) + \varepsilon\,\dot{s}(t_1) + k'(0)\varepsilon\,\eta(t_1) - s(t_1)\right]$$

Thus

$$\ddot{s}(t_1) + k'(0)\,\dot{\eta}(t_1) = \left(\frac{d\phi}{ds_1}\right)_{s_1}\left[\dot{s}(t_1) + k'(0)\,\eta(t_1)\right]$$

Or

$$\boxed{\left[\left(\frac{d\phi}{ds_1}\right)_{s_1}\eta(t_1) - \dot{\eta}(t_1)\right]k'(0) = \ddot{s}(t_1) - \left(\frac{d\phi}{ds_1}\right)_{s_1}\dot{s}(t_1)} \qquad (11)$$

This determines $k'(0) = \left(\frac{dk}{d\varepsilon}\right)_{\varepsilon=0}$

Let us write

$$W(s,\dot{s})\, e^{\frac{1}{c}(\dot{s}+gt)} = \mathcal{F}(t, s, \dot{s}) \qquad (12)$$

By substituting (9), (10), (12) into (5), M_0 can be considered as a function of ε (s, η specified). Thus

$$M_0(\varepsilon) = \frac{1}{c}\int_0^{t_1+\varepsilon} \mathcal{F}(t,\ s + h(\varepsilon)\eta,\ \dot{s} + h(\varepsilon)\dot{\eta})\, dt$$

$$+ M_1 e^{\frac{1}{c}\{\dot{s}(t_1) + \varepsilon\ddot{s}(t_1) + h(\varepsilon)\dot{\eta}(t_1) + gt_1 + g\varepsilon\}}$$

The condition that s be the desired function requires that $\frac{\partial M_0}{\partial \varepsilon} = 0$ at $\varepsilon = 0$. But

$$\left(\frac{\partial M_0}{\partial \varepsilon}\right)_{\varepsilon=0} = \frac{1}{c}\mathcal{F}(t_1, s_1, \dot{s}_1) + \frac{1}{c}h'(0)\int_0^{t_1}\left[\eta\frac{\partial \mathcal{F}}{\partial s} + \dot{\eta}\frac{\partial \mathcal{F}}{\partial \dot{s}}\right]dt$$

$$+ M_1 e^{\frac{1}{c}\{\dot{s}_1 + gt_1\}}\frac{1}{c}\left[\ddot{s}(t_1) + h'(0)\dot{\eta}(t_1) + g\right]$$

$$= \frac{1}{c}h'(0)\int_0^{t_1}\eta\left[\frac{\partial \mathcal{F}}{\partial s} - \frac{d}{dt}\left(\frac{\partial \mathcal{F}}{\partial \dot{s}}\right)\right]dt + \frac{1}{c}h'(0)\eta(t_1)\left(\frac{\partial \mathcal{F}}{\partial \dot{s}}\right)_{t=t_1}$$

$$+ \frac{1}{c}\mathcal{F}(t_1, s_1, \dot{s}_1) + \frac{1}{c}M_1 e^{\frac{1}{c}(\dot{s}_1+gt_1)}\left[(\ddot{s}_1 + g) + h'(0)\dot{\eta}(t_1)\right]$$

Aside from the condition that $\eta(0) = 0$, the function η is arbitrary. Therefore, in order the above expression be zero,

$$\boxed{\frac{\partial \mathcal{F}}{\partial s} - \frac{d}{dt}\left(\frac{\partial \mathcal{F}}{\partial \dot{s}}\right) = 0} \qquad (13)$$

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$$0 = k'(0)\eta(t_1)\left(\frac{\partial \mathcal{F}}{\partial \dot{s}}\right)_{t=t_1} + \mathcal{F}(t_1, s_1, \dot{s}_1) + M_1 e^{\frac{1}{c}(\dot{s}_1 + gt_1)}\left[(\ddot{s}_1 + g) + k'(0)\dot{\eta}(t_1)\right]$$

Multiply this equation by $\left[\left(\frac{d\phi}{ds_1}\right)_{s_1}\eta(t_1) - \dot{\eta}(t_1)\right]$ and then use (11), we have

$$\left[\ddot{s}_1 - \left(\frac{d\phi}{ds_1}\right)_{s_1}\dot{s}_1\right]\eta(t_1)\left(\frac{\partial \mathcal{F}}{\partial \dot{s}}\right)_{t=t_1} + \left[\left(\frac{d\phi}{ds_1}\right)_{s_1}\eta(t_1) - \dot{\eta}(t_1)\right]\mathcal{F}(t_1, s_1, \dot{s}_1)$$

$$+ M_1 e^{\frac{1}{c}(\dot{s}_1 + gt_1)}\left[\left\{\left(\frac{d\phi}{ds_1}\right)_{s_1}\eta(t_1) - \dot{\eta}(t_1)\right\}(\ddot{s}_1 + g) + \dot{\eta}(t_1)\left\{\ddot{s}_1 - \left(\frac{d\phi}{ds_1}\right)_{s_1}\dot{s}_1\right\}\right] = 0.$$

Since η is arbitrary, $\eta(t_1)$ and $\dot{\eta}(t_1)$ are also arbitrary, hence for the above relation to be true, we have following two equations,

$$\left\{\ddot{s}_1 - \left(\frac{d\phi}{ds_1}\right)_{s_1}\dot{s}_1\right\}\left(\frac{\partial \mathcal{F}}{\partial \dot{s}}\right)_{t=t_1} + \left(\frac{d\phi}{ds_1}\right)_{s_1}\mathcal{F}(t_1, s_1, \dot{s}_1) + M_1 e^{\frac{1}{c}(\dot{s}_1 + gt_1)}\left(\frac{d\phi}{ds_1}\right)_{s_1}(\ddot{s}_1 + g) = 0$$

and

$$\mathcal{F}(t_1, s_1, \dot{s}_1) + M_1 e^{\frac{1}{c}(\dot{s}_1 + gt_1)}\left[\ddot{s}_1 + g - \ddot{s}_1 + \left(\frac{d\phi}{ds_1}\right)_{s_1}\dot{s}_1\right] = 0.$$

These equations can be put into more convenient forms by using (12).

$$\mathcal{F}(t_1, s_1, \dot{s}_1) = e^{\frac{1}{c}(\dot{s}_1 + gt_1)}\, W(s_1, \dot{s}_1)$$

$$\left(\frac{\partial \mathcal{F}}{\partial \dot{s}}\right)_{t=t_1} = e^{\frac{1}{c}(\dot{s}_1 + gt_1)}\left[\left(\frac{\partial W}{\partial \dot{s}}\right)_1 + \frac{W_1}{c}\right]$$

Thus

$$\left\{\ddot{s}_1 - \left(\frac{d\phi}{ds_1}\right)\dot{s}_1\right\}\left\{\left(\frac{\partial W}{\partial \dot{s}}\right)_1 + \frac{W_1}{c}\right\} + \left(\frac{d\phi}{ds_1}\right)W_1 + M_1\left(\frac{d\phi}{ds_1}\right)(\ddot{s}_1 + g) = 0 \quad (14)$$

$$W_1 + M_1\left[\left(\frac{d\phi}{ds_1}\right)\dot{s}_1 + g\right] = 0 \quad (15)$$

(6) gives the relation between flight velocity and altitude during coasting. During coasting $dH/dt=0$, so (6) gives

$$\left(\frac{d\phi}{dt}+g\right)M_1 + W_1 = 0.$$

Or

$$\left(\frac{d\phi}{ds_1}\dot{s}_1 + g\right)M_1 + W_1 = 0$$

Therefore (15) is automatically satisfied if s_1 and $\dot{s}_1$ satisfies (6). (14) can be written as

$$\ddot{s}_1\left\{\left(\frac{\partial W}{\partial \dot{s}}\right)_1 + \frac{W_1}{c}\right\} + \left(\frac{d\phi}{ds_1}\right)\left[-\dot{s}\left\{\left(\frac{\partial W}{\partial \dot{s}}\right)_1 + \frac{W_1}{c}\right\} + W_1 + M_1(\ddot{s}_1 + g)\right] = 0$$

But (15) gives

$$\left(\frac{d\phi}{ds_1}\right) = -\frac{W_1 + M_1 g}{M_1 \dot{s}_1}$$

Hence

$$\boxed{M_1\dot{s}_1\ddot{s}_1\left\{\left(\frac{\partial W}{\partial \dot{s}}\right)_1 + \frac{W_1}{c}\right\} = (W_1 + M_1 g)\left[-\dot{s}_1\left\{\left(\frac{\partial W}{\partial \dot{s}}\right)_1 + \frac{W_1}{c}\right\} + W_1 + M_1(\ddot{s}_1 + g)\right]} \tag{16}$$

<u>This is the condition at $t=t_1$ for the extremal</u>. Together with the condition that $s(0)=0$, it completely and uniquely determines $s(t)$.

6

Now specialize to

$$W = W_0 e^{-\alpha s} \dot{s}^2 \tag{17}$$

$$\frac{\partial W}{\partial \dot{s}} = 2 W_0 e^{-\alpha s} \dot{s}$$

Then (11) becomes

$$W_0 M_1 e^{-\alpha s_1} \dot{s}_1^2 \ddot{s}_1 \left(2 + \frac{\dot{s}_1}{c} \right) = \left[W_0 e^{-\alpha s_1} \dot{s}_1^2 + M_1 g \right] \left[-W_0 e^{-\alpha s_1} \dot{s}_1^2 \left(1 + \frac{\dot{s}_1}{c} \right) + M_1 (\ddot{s}_1 + g) \right]$$

3.2

The Transfer Functions of Rocket Nozzles

火箭喷管的传递函数

这是作者发表于1952年的“The Transfer Functions of Rocket Nozzles”（火箭喷管的传递函数）一文的部分手稿，共有15页。

作者讨论的是一个有关火箭发动机的燃烧稳定性的问题。当时有人做过这方面的研究，他们假设燃烧室内压力增加的百分数与通过燃烧室喷管的质量流率增加的百分数是相等的。作者怀疑这一假设是否成立，便用一维气体动力学的模型，仔细考察了质量流率的相对变化与压力的相对变化之间的比值究竟存在什么样的关系。这一比值便称之为火箭喷管的传递函数。假设燃烧室内的流动参数（如压力、密度、流速等）在平均值附近发生振荡，作者的研究结果是，这一比值不是常数而与振荡频率有关。只是在频率很低的情况下，前人的假设才近似成立；而在频率很高的情况下，这个比值则近似等于 $1+(\gamma \cdot M_1)^{-1}$，其中 γ 是气体的比热比，而 M_1 是喷管进口处的马赫数。

这里选印了手稿中的两个部分。一部分选自作者认为不满意而放弃的部分推导手稿，选了其中的3页。注意作者在第1页的标题前用红笔注上“Not Correct！”（不正确！）；并在第16页上用红笔打了问号，作者在这里已经发现了问题。另一部分是为发表论文而撰写的初始底稿，共12页。从两部分的对比中可以看到，作者发现喷管出口处的情况相当复杂，必须要考虑喷出的射流与周围空气的相互作用，而前面之所以“不正确”就是因为忽略了周围空气的影响。

1

Not exact!

Impedance of a De Laval Nozzle.

1) Exit jet

φ = disturbance velocity potential

$$\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{\partial^2\varphi}{\partial x^2}=\frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2}+\frac{2M}{a}\frac{\partial^2\varphi}{\partial x\partial t}+M^2\frac{\partial^2\varphi}{\partial x^2}$$

Let $\varphi = f(r)\sin\omega(t-\frac{x}{c}+\alpha)$

$$\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}-\left(\frac{\omega}{c}\right)^2 f=-\left(\frac{\omega}{a}\right)^2 f+2\frac{M\omega}{a}\frac{\omega}{c}f-M^2\left(\frac{\omega}{c}\right)^2 f$$

$$\boxed{\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}+\left[\left(\frac{\omega}{a}-M\frac{\omega}{c}\right)^2-\left(\frac{\omega}{c}\right)^2\right]f=0}$$

$$f = A\,J_0\left(\sqrt{\left(\frac{\omega}{a}-M\frac{\omega}{c}\right)^2-\left(\frac{\omega}{c}\right)^2}\;r\right)$$

Let R = radius of jet, then

$$J_0\left(\sqrt{\left(\frac{\omega}{a}-M\frac{\omega}{c}\right)^2-\left(\frac{\omega}{c}\right)^2}\;R\right)=0$$

If λ is the first root of J_0, then

$$\boxed{\left[\left(\frac{\omega}{a}-M\frac{\omega}{c}\right)^2-\left(\frac{\omega}{c}\right)^2\right]R^2=\lambda^2}$$

But $\frac{\partial\varphi}{\partial t}+Uu'+\frac{p'}{\rho}=0$

Or $\frac{\partial\varphi}{\partial t}+U\frac{\partial\varphi}{\partial x}+\frac{\bar{p}}{\bar{\rho}}\gamma\frac{\rho'}{\bar{\rho}}=0$

$$\frac{\partial\varphi}{\partial t}+U\frac{\partial\varphi}{\partial x}+a^2\frac{\rho'}{\rho}=0$$

2

Or $$a^2\frac{\rho'}{\rho} = -\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x}\right)$$

$$= -\omega f(r)\cos\omega\left(t - \frac{x}{c} + \alpha\right)\left[1 - \frac{u}{c}\right]$$

$$= -\omega c\left(1 - \frac{u}{c}\right) f(r)\cos\omega\left(t - \frac{x}{c} + \alpha\right)\cdot\frac{1}{c}$$

$$= (c - u)u' = u(c - u)\left(\frac{u'}{u}\right)$$

Thus at exit $$\boxed{\frac{\rho'}{\rho} = M_e^2\left(\frac{c}{u_e} - 1\right)\left(\frac{u'}{u}\right)}$$

$$\left(\frac{\omega}{a}\frac{R}{\lambda}\right)^2\left[\left(1 - \frac{u_e}{c}\right)^2 - \left(\frac{a}{c}\right)^2\right] = 1$$

$$\left(1 - \frac{u_e}{c}\right)^2 - \frac{1}{M_e^2}\left(\frac{u_e}{c}\right)^2 = \left(\frac{a\lambda}{\omega R}\right)^2$$

$$1 - 2\frac{u_e}{c} + \left(1 - \frac{1}{M_e^2}\right)\left(\frac{u_e}{c}\right)^2 = \left(\frac{a\lambda}{\omega R}\right)^2$$

$$\left[1 - \left(\frac{a\lambda}{\omega R}\right)^2\right]\left(\frac{c}{u_e}\right)^2 - 2\left(\frac{c}{u_e}\right) + \left(1 - \frac{1}{M_e^2}\right) = 0$$

$$\boxed{\frac{c}{u_e} = \frac{1}{\left[1 - \left(\frac{a\lambda}{\omega R}\right)^2\right]}\left\{1 \pm \sqrt{1 - \left(1 - \frac{1}{M_e^2}\right)\left[1 - \left(\frac{a\lambda}{\omega R}\right)^2\right]}\right\}}$$

$c > 0$. So if $\frac{a\lambda}{\omega R} < 1$, both root are possible

if $\frac{a\lambda}{\omega R} > 1$, then

$$\frac{c}{u_e} = \frac{1}{\left[\left(\frac{a\lambda}{\omega R}\right)^2 - 1\right]}\left\{\sqrt{1 + \left(1 - \frac{1}{M_e^2}\right)\left[\left(\frac{a\lambda}{\omega R}\right)^2 - 1\right]} - 1\right\}$$

11

$$\frac{\bar{\rho}\left(1+\frac{\rho'}{\bar{\rho}}\right)\bar{u}\left(1+\frac{u'}{\bar{u}}\right)-\bar{\rho}\bar{u}}{\bar{\rho}\bar{u}} \cong \frac{\rho'}{\bar{\rho}}+\frac{u'}{\bar{u}}$$

$$\frac{p'}{\bar{p}}=\gamma\frac{\rho'}{\bar{\rho}}$$

$$\mu=\left(\frac{\dot{m}'}{\bar{\dot{m}}}\Big/\frac{p'}{\bar{p}}\right)_0=\frac{F(0)+G(0)}{\gamma F(0)}=\frac{1}{\gamma}\left[1+\frac{G(0)}{F(0)}\right]$$

$$\mu^0=\frac{1}{\gamma}\left[1+\frac{G(0)}{F(0)}\right]=\frac{1}{\gamma}\frac{\frac{M_1^2-1}{2-M_1^2}}{\frac{M_1^2-1}{1-\frac{M_1^2}{2}}+\left(\qquad\right)^{\frac{1}{\gamma+1}}}$$

$$\boxed{\mu^0=\frac{1}{\gamma}\frac{1}{2-\left(\frac{M_1^2-2}{M_1^2-1}\right)e^{i\frac{\pi}{\gamma+1}}\left(\frac{M_1^2-1}{1-M_0^2}\,\frac{1+\frac{\gamma-1}{2}M_0^2}{1+\frac{\gamma-1}{2}M_1^2}\right)^{\frac{1}{\gamma+1}}}}$$

?

Let $M_0=0,\quad M_1\to\infty$

$$\mu^0=\frac{1}{\gamma}\frac{1}{2-\left(\frac{2}{\gamma-1}\right)^{\frac{1}{\gamma+1}}e^{i\frac{\pi}{\gamma+1}}}=\frac{1}{\gamma}\frac{1}{\left[2-\left(\frac{2}{\gamma-1}\right)^{\frac{1}{\gamma+1}}\cos\left(\frac{\pi}{\gamma+1}\right)\right]-i\left(\frac{2}{\gamma-1}\right)^{\frac{1}{\gamma+1}}\sin\left(\frac{\pi}{\gamma+1}\right)}$$

$$=\frac{1}{\gamma}\frac{\left[2-\left(\frac{2}{\gamma-1}\right)^{\frac{1}{\gamma+1}}\cos\left(\frac{\pi}{\gamma+1}\right)\right]+i\left(\frac{2}{\gamma-1}\right)^{\frac{1}{\gamma+1}}\sin\left(\frac{\pi}{\gamma+1}\right)}{2-4\left(\frac{2}{\gamma-1}\right)^{\frac{1}{\gamma+1}}\cos\left(\frac{\pi}{\gamma+1}\right)+\left(\frac{2}{\gamma-1}\right)^{\frac{2}{\gamma+1}}}$$

The Hydraulic Impedance of a De Laval Nozzle

Recently, the problem of combustion instability of rocket motor has been studied by several authors (Refs. 1, 2 and 3). In these investigations, it is ~~usually~~ assumed that the percentage increase of the mass rate of flow through the nozzle is equal to the percentage increase of pressure in the rocket cylinder. It is however not certain whether this assumption is correct. Since the flow conditions enter in a direct manner into the instability calculation, the relation between flow variations and the pressure variations, i.e., the hydraulic impedance of the nozzle, should be determined more carefully. It is the purpose of this paper to do this. The result of the present study indicates that the hydraulic impedance of a De Laval nozzle is a rather complex function of the nozzle geometry and the frequency of oscillation and the previous very simple assumption is not justified.

End Conditions

The flow in the nozzle will be considered as one dimensional, i.e., at each nozzle section, the conditions are taken to be uniform and the only independent variables of the problem are the time t and the distance x along the nozzle axis. Let p be the pressure, ρ the density, and u the velocity. The primed quantities are the fluctuating quantities, thus ρ' is the density fluctuations. Similarly the unprimed quantities are the steady state or undisturbed quantities. Therefore p'/p is the fractional oscillating pressure in terms of the steady state pressure. (These fractional quantities are assumed to be small so that only first order terms are considered.) Hence the purpose of this paper can now be stated as simply to compute the vector $\left(\frac{\rho'}{\rho}+\frac{u'}{u}\right)\Big/\left(\frac{p'}{p}\right)$ at the entrance to the nozzle.

The conditions at the entrance to the nozzle is fixed by the

plausible assumption that the temperature of the combustion gas is not changed by variations in pressure. Let the gas be considered as a perfect gas, and let the subscript 1 denote the entrance to the nozzle, Then

$$\left(\frac{p'}{p}\right)_1 - \left(\frac{\rho'}{\rho}\right)_1 = 0 \tag{1}$$

At the exit of the nozzle, the situation is more complicated for one must consider the interaction of the exit jet with the surrounding air. For this purpose, it would be necessary to introduce another independent variable r to denote the distance from the jet axis and another dependent variable v' to denote velocity in the direction of r. Then using the subscript 2 to denote quantities at the exit of the nozzle, which are used to represent those in the jet as mixing of the jet and the surrounding air is not considered, one has the following linearized equations of motion in the jet,

$$\frac{\partial}{\partial t}\left(\frac{\rho'}{\rho_2}\right) + u_2\frac{\partial}{\partial x}\left(\frac{\rho'}{\rho_2}\right) + u_2\frac{\partial}{\partial x}\left(\frac{u'}{u_2}\right) + u_2\frac{\partial}{\partial r}\left(\frac{v'}{u_2}\right) + u_2\frac{1}{r}\left(\frac{v'}{u_2}\right) = 0 \tag{2}$$

$$\frac{\rho_2 u_2}{p_2}\left\{\frac{\partial}{\partial t}\left(\frac{u'}{u_2}\right) + u_2\frac{\partial}{\partial x}\left(\frac{u'}{u_2}\right)\right\} = -\frac{\partial}{\partial x}\left(\frac{p'}{p_2}\right) \tag{3}$$

$$\frac{\rho_2 u_2}{p_2}\left\{\frac{\partial}{\partial t}\left(\frac{v'}{u_2}\right) + u_2\frac{\partial}{\partial x}\left(\frac{v'}{u_2}\right)\right\} = -\frac{\partial}{\partial r}\left(\frac{p'}{p_2}\right) \tag{4}$$

A relation between (p'/p_2) and (ρ'/ρ_2) is still needed. This is supplied by noting that if there are pressure and density fluctuations in the rocket cylinder but no temperature fluctuations there, the entropy of the gas must also fluctuates. The entropy fluctuations are known to be carried along the gas. Thus if ε is the amplitude of entropy oscillation and ω the angular frequency of the oscillation,

$$\frac{p'}{p_2} - \gamma \frac{\rho'}{\rho_2} = \epsilon \sin\omega\left(t - \frac{x}{u_2} + \alpha\right) \tag{5}$$

where α is an arbitrary phase shift.

The appropriate solutions are then

$$\frac{\rho'}{\rho_2} = f(r)\sin\omega\left(t - \frac{x}{u_2} + \alpha\right) \tag{6}$$

$$\frac{u'}{u_2} = g(r)\sin\omega\left(t - \frac{x}{u_2} + \alpha\right) \tag{7}$$

$$\frac{v'}{u_2} = h(r)\cos\omega\left(t - \frac{x}{u_2} + \alpha\right) \tag{8}$$

By substituting Eqs. (6) to (8) into the equations of motion, it is found that

$$\gamma f(r) + \epsilon = 0$$

$$\frac{df}{dr} = 0$$

Therefore

$$f(r) = \text{constant} = -\frac{\epsilon}{\gamma} \tag{9}$$

But then Eq (5) shows that the pressure fluctuations p'/p_2 in the jet must be zero. The entropy wave then manifests itself as a density wave in the jet. The zero amplitude of pressure fluctuation in the jet and hence at exit of the nozzle is the second end condition sought for:*

$$\left(\frac{p'}{p}\right)_2 = 0 \tag{10}$$

<u>Formulation of the Problem in Nozzle</u>

In the nozzle, the linearized continuity equation for the oscillating quantities is

$$\frac{\partial}{\partial t}\left(\frac{\rho'}{\rho}\right) + u\frac{\partial}{\partial x}\left(\frac{\rho'}{\rho} + \frac{u'}{u}\right) = 0 \tag{11}$$

The dynamic equation is

* The interaction of jet with the surrounding air is discussed in the Appendix.

$$\frac{\partial}{\partial t}\left(\frac{u'}{u}\right)+\frac{du}{dx}\left(\frac{\rho'}{\rho}+2\frac{u'}{u}\right)+u\frac{\partial}{\partial x}\left(\frac{u'}{u}\right)=\frac{du}{dx}\left(\frac{p'}{p}\right)-\frac{p}{\rho u^2}u\frac{\partial}{\partial x}\left(\frac{p'}{p}\right) \quad (12)$$

The equation for constant entropy of any fluid mass is then

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)\left[\frac{p'}{p}-\gamma\frac{\rho'}{\rho}\right]=0 \quad (13)$$

It is convenient in the following calculations to introduce a specific nozzle shape such that the steady velocity in the nozzle increase linearly with x. The simplest way to specify this is

$$u=\frac{u_2}{x_2}x \quad (14)$$

The origin of the x-axis is not generally at the entrance to the nozzle, it is there only if the steady velocity at entrance is equal to zero. With Eq. (14), Eq. (13) becomes

$$\left(\frac{\partial}{\partial t}+\frac{u_2}{x_2}\frac{\partial}{\partial(\log x)}\right)\left[\frac{p'}{p}-\gamma\frac{\rho'}{\rho}\right]=0$$

Therefore if the entropy oscillation at the entrance to the nozzle is

$$\left(\frac{p'}{p}\right)_1-\gamma\left(\frac{\rho'}{\rho}\right)_1=\varepsilon\sin\omega t \quad (15)$$

then in general

$$\left(\frac{p'}{p}\right)-\gamma\left(\frac{\rho'}{\rho}\right)=\varepsilon\sin\omega\left(t-\frac{x_2}{u_2}\log\frac{x}{x_1}\right) \quad (16)$$

By eliminating (p'/p) between Eqs. (12) and (16), the resultant equation together with Eq. (11) constitute a system of two equations for the two unknowns (ρ'/ρ) and (u'/u).

Now introduce the nondimensional parameters

$$\xi=\frac{x}{x_2} \quad (17)$$

$$\beta=\frac{\omega x_2}{u_2} \quad (18)$$

and let

$$\left(\frac{\rho'}{\rho}\right) = f_1(\xi)\sin\omega t + f_2(\xi)\cos\omega t \qquad (19)$$

$$\left(\frac{u'}{u}\right) = g_1(\xi)\sin\omega t + g_2(\xi)\cos\omega t \qquad (20)$$

Then the equations for the f's and the g's are

$$-\beta f_2(\xi) + \xi\left[f_1'(\xi) + g_1'(\xi)\right] = 0$$

$$\beta f_1(\xi) + \xi\left[f_2'(\xi) + g_2'(\xi)\right] = 0$$

$$-\beta g_2(\xi) + \left[f_1(\xi) + 2g_1(\xi)\right] + \xi g_1'(\xi) = \gamma f_1(\xi) - \frac{1}{\gamma M^2}\xi f_1'(\xi) + \varepsilon\cos\beta\left(\log\frac{\xi}{\xi_1}\right) + \frac{\beta}{\gamma M^2}\varepsilon\sin\beta\left(\log\frac{\xi}{\xi_1}\right)$$

$$\beta g_1(\xi) + \left[f_2(\xi) + 2g_2(\xi)\right] + \xi g_2'(\xi) = \gamma f_2(\xi) - \frac{1}{\gamma M^2}\xi f_2'(\xi) - \varepsilon\sin\beta\left(\log\frac{\xi}{\xi_1}\right) + \frac{\beta}{\gamma M^2}\varepsilon\cos\beta\left(\log\frac{\xi}{\xi_1}\right)$$

where M is the local Mach number of the steady flow, a function of ξ and

$$\xi_1 = \frac{x_1}{x_2} \qquad (21)$$

The equations for f and g can be simplified by using complex quantities:

$$F(\xi) = f_1(\xi) + i f_2(\xi) \qquad (22)$$

$$G(\xi) = g_1(\xi) + i g_2(\xi) \qquad (23)$$

Then

$$\xi\left[F'(\xi) + G'(\xi)\right] + i\beta F(\xi) = 0 \qquad (24)$$

and $$(2+i\beta)G(\xi) + (1-i\beta)F(\xi) - \xi F'(\xi) = \gamma F(\xi) - \frac{1}{\gamma M^2}\xi F'(\xi) + \varepsilon\left(\frac{\xi}{\xi_1}\right)^{-i\beta}\left[1 + \frac{i\beta}{\gamma M^2}\right] \qquad (25)$$

Now the Mach number M can be expressed in terms of x, in fact

$$M^2 = \frac{M_2^2\xi^2}{1 + \frac{\gamma-1}{2}M_2^2(1-\xi^2)} \qquad (26)$$

where M_2 is the Mach number of the steady nozzle flow at the exit and is here much larger than unity.

By eliminating $G(\xi)$ from Eqs. (24) and (25) a single second order equation for $F(\xi)$ is obtained. The result can, however, be reduced to more convenient form by using a new independent variable z defined as

$$z = \frac{\frac{\gamma-1}{2} M_2^2 \xi^2}{1 + \frac{\gamma-1}{2} M_2^2} \tag{27}$$

It is easy to show that z is actually the square of the ratio of u to the so-called critical sound speed. Thus $z = 1$ at the throat of the de Laval nozzle. In terms of z, the differential equation for F is

$$z(1-z)\frac{d^2F}{dz^2} - \left[2 + \frac{2i\beta}{\gamma+1}\right] z \frac{dF}{dz} - \frac{i\beta(3+i\beta)}{2(\gamma+1)} F$$
$$= -i\beta\varepsilon\left(\frac{z}{z_1}\right)^{-\frac{i\beta}{2}} \left[\frac{1 - i\beta\frac{\gamma-1}{2\gamma}}{2(\gamma+1)} + \frac{3+i\beta}{4\gamma}\frac{1}{z}\right] \tag{28}$$

The relation between $F(z)$ and $G(z)$ is

$$(3+i\beta)G(z) = [(\gamma-1)+i\beta]F(z) - (\gamma+1)(1-z)\frac{dF}{dz} + \varepsilon\left(\frac{z}{z_1}\right)^{-i\beta/2}\left[1 - \frac{i\beta(\gamma-1)}{2\gamma} + \frac{i\beta(\gamma+1)}{2\gamma}\frac{1}{z}\right] \tag{29}$$

z_1 is of course the value of z corresponding to ξ_1 as given by Eq. (27), i.e.,

$$z_1 = \frac{\frac{\gamma-1}{2} M_2^2 \xi_1^2}{1 + \frac{\gamma-1}{2} M_2^2} \tag{30}$$

The end conditions needed to solve Eq. (28) are as follows: At $x = x_1$ or $z = z_1$, Eq. (11) has to be satisfied together with Eq. (15). Therefore

$$F(z_1) = -\frac{\varepsilon}{\gamma-1} \tag{31}$$

At

$$z_2 = \frac{\frac{\gamma-1}{2} M_2^2}{1 + \frac{\gamma-1}{2} M_2^2} \tag{32}$$

Eqs. (13) and (16) have to be satisfied. Hence

$$F(z_1) = -\frac{1}{\gamma}\varepsilon(\xi_1)^{i\beta} \tag{33}$$

By knowing $F(z)$, one can compute $G(z)$ by Eq. (29). Then the density oscillations and the velocity oscillation are determined as functions proportional to the amplitude of entropy oscillations ε. Since the point of interest is the ratio of the oscillations, the arbitrary ε does not really enter into the final result.

<u>Solution for small β</u>

Although Eq. (28) is the hypergeometric differential equation having straightforward method of solution, the calculations involved are nevertheless very tedious. For a lucid treatment, only cases of either very small frequencies or very large frequencies will be considered.

If the frequency β is very small, the functions F and G can be expanded in terms of this parameter:

$$F(z;\beta) = F^{(0)}(z) + \beta F^{(1)}(z) + \cdots \tag{34}$$

$$G(z;\beta) = G^{(0)}(z) + \beta G^{(1)}(z) + \cdots \tag{35}$$

By substituting these expressions into Eqs. (28) and (29) and equating terms of equal powers in β, one has

$$z(1-z)\frac{d^2F^{(0)}}{dz^2} - 2z\frac{dF^{(0)}}{dz} = 0 \tag{36}$$

and

$$z(1-z)\frac{d^2F^{(1)}}{dz^2} - 2z\frac{dF^{(1)}}{dz} = i\left\{\frac{2}{\gamma+1}z\frac{dF^{(0)}}{dz} + \frac{1}{\gamma+1}F^{(0)} - \varepsilon\right\} \tag{37}$$

$$2G^{(0)}(z) = (\gamma-1)F^{(0)}(z) - (\gamma+1)(1-z)\frac{dF^{(0)}}{dz} + \varepsilon \tag{38}$$

and

$$2G^{(1)}(z) = (\gamma-1)F^{(1)}(z) - (\gamma+1)(1-z)\frac{dF^{(1)}}{dz} + i\left[F^{(0)}(z) - G^{(0)}(z)\right] - \frac{i\varepsilon}{2}\log\left(\frac{z}{z_1}\right) \tag{39}$$

The end conditions (31) and (33) now give

$$F^{(0)}(z_1) = -\frac{\varepsilon}{\gamma-1}\ ; \quad F^{(1)}(z_1) = 0 \tag{40}$$

and

$$F^{(0)}(z_2) = -\frac{\varepsilon}{\gamma}\ ; \quad F^{(1)}(z_2) = -\frac{\varepsilon}{\gamma}\, i \log \xi_1 \tag{41}$$

The solution of the equations is now quite easy. There is a singular point at $z = 1$, of course. But both $F^{(0)}$ and $F^{(1)}$ are expressible in elementary functions. The most important result is however the values of $G^{(0)}$ and $G^{(1)}$ at the entrance $z = z_1$, thus only these will be explicitly given here.

$$G^{(0)}(z_1)/F(z_1) = -\frac{1}{\gamma}\frac{M_2^2-1}{M_2^2}\frac{1}{1-\xi_1^2} \tag{42}$$

$$\begin{aligned} G^{(1)}(z_1)/F(z_1) = i \Bigg[& \frac{\gamma-1}{2\gamma}\frac{M_2^2-1}{M_2^2}\frac{1}{1-\xi_1^2}\log\xi_1^2 + \frac{1}{2}\left\{1+\frac{1}{\gamma}\frac{M_2^2-1}{M_2^2}\frac{1}{1-\xi_1^2}\right\} \\ & + \frac{1}{2\gamma}\frac{M_2^2-1}{1+\frac{\gamma-1}{2}M_2^2}\left\{\frac{2}{\gamma+1}\frac{M_2^2-1}{M_2^2\, [illegible]}-1\right\}\Bigg\{\frac{\log\xi_1^2}{1-\xi_1^2} - \log\left(1-\frac{\frac{\gamma-1}{2}M_2^2\xi_1^2}{1+\frac{\gamma-1}{2}M_2^2}\right) \\ & + \frac{1+\gamma M_2^2}{\frac{\gamma-1}{2}M_2^2}\frac{1}{1-\xi_1^2}\log\left(\frac{M_2^2-1}{1+\frac{\gamma-1}{2}M_2^2}\right) + \left(1+\xi_1^2-\frac{M_2^2-1}{\frac{\gamma-1}{2}M_2^2}\right)\frac{1}{1-\xi_1^2}\log\frac{1+\frac{\gamma-1}{2}M_2^2}{1-M_2^2+\frac{\gamma-1}{2}M_2^2(1-\xi_1^2)}\Bigg\} \\ & + \frac{1}{2}\left\{\frac{2}{\gamma(\gamma+1)}\frac{M_2^2-1}{M_2^2}\frac{1}{1-\xi_1^2}-\gamma^2\right\}\left\{\frac{1}{1-\xi_1^2}\log\xi_1^2+1\right\}\Bigg] \end{aligned} \tag{43}$$

It is quite clear from these results that since $G(z_1) \neq 0$, the fractional increase in mass rate of flow cannot be equal to the fractional increase in pressure. The correct magnitude depends upon the entrance Mach number, the exit Mach number, γ and the frequency of oscillation. In fact, the ratio of fractional increase in mass rate of flow and that of pressure is

$$\left[\left(\frac{\rho'}{\rho}\right)_1 + \left(\frac{u'}{u}\right)_1\right] \Big/ \left(\frac{p'}{p}\right)_1 = 1 + \frac{G^{(0)}(z_1)}{F(z_1)} + \beta\,\frac{G^{(1)}(z_1)}{F(z_1)} + \cdots \tag{44}$$

The discussion of the numerical results will be postponed till a later section.

<u>Solution for large β</u>

If the value of β is very large, the dominating terms in Eq. ([illegible]) are

$$z(1-z)\frac{d^2F}{dz^2} - \frac{2i\beta}{\gamma+1} z \frac{dF}{dz} + \frac{\beta^2}{2(\gamma+1)} F = \beta^2 \varepsilon \left(\frac{z}{z_1}\right)^{-\frac{i\beta}{2}} \frac{1}{4\gamma}\left[\frac{1}{z} - \frac{\gamma-1}{\gamma+1}\right] \tag{45}$$

For the particular integral F^*, take

$$F^*(z) = Z(z)\left(\frac{z}{z_1}\right)^{-\frac{i\beta}{2}}$$

where $Z(z)$ is a function of z not involving β. Therefore by retaining only the highest order terms,

$$\frac{dF^*}{dz} \cong -\frac{i\beta}{2} \frac{Z(z)}{z}\left(\frac{z}{z_1}\right)^{-\frac{i\beta}{2}} \tag{46}$$

$$\frac{d^2F^*}{dz^2} \cong -\frac{\beta^2}{4} \frac{Z(z)}{z^2}\left(\frac{z}{z_1}\right)^{-\frac{i\beta}{2}}$$

By substituting these derivatives into Eq. (45), it is found that

$$Z(z) = -\frac{\varepsilon}{\gamma}$$

Thus

$$F^*(z) = -\frac{\varepsilon}{\gamma}\left(\frac{z}{z_1}\right)^{-\frac{i\beta}{2}} \tag{47}$$

To find the complementary function, let

$$F(z) = e^{i\beta\lambda(z)}$$

Then

$$\frac{dF}{dz} = i\beta e^{i\beta\lambda(z)} \frac{d\lambda}{dz} \tag{48}$$

$$\frac{d^2F}{dz^2} \cong -\beta^2 e^{i\beta\lambda(z)}\left(\frac{d\lambda}{dz}\right)^2$$

By substituting these into the homogeneous equation corresponding to Eq. (45), one has

$$\frac{d\lambda_{1,2}}{dz} = \frac{1}{(\gamma+1)(1-z)}\left[1 \pm \sqrt{1 + \frac{\gamma+1}{2}\frac{1-z}{z}}\right] \tag{49}$$

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where
$$\lambda_1(z) = \frac{1}{(\gamma+1)}\int_{z_1}^{z} \frac{dz}{1-z}\left[1+\sqrt{1+\frac{\gamma+1}{2}\frac{1-z}{z}}\right], \quad \lambda_1(z_1)=0 \tag{50}$$

$$\lambda_2(z) = \frac{1}{(\gamma+1)}\int_{z_1}^{z} \frac{dz}{1-z}\left[1-\sqrt{1+\frac{\gamma+1}{2}\frac{1-z}{z}}\right], \quad \lambda_2(z_1)=0 \tag{51}$$

The complete solution for very large β is then
$$F(z) = A\,e^{i\beta\lambda_1(z)} + B\,e^{i\beta\lambda_2(z)} - \frac{\varepsilon}{\gamma}\left(\frac{z}{z_1}\right)^{-\frac{i\beta}{2}} \tag{52}$$
where A and B are two constants to be determined by the end conditions. To satisfy the end conditions as specified by Eqs. (31) and (33),
$$A+B = -\varepsilon\frac{1}{\gamma(\gamma-1)}$$
and
$$A e^{i\beta\lambda_1(z_2)} + B e^{i\beta\lambda_2(z_2)} = 0$$

These equations then determine the constants A and B.

Here again, only the final results for $G(z_1)$ will be explicitly given:
$$G(z_1)/F(z_1) = -\frac{i}{\gamma}\sqrt{1+\frac{\gamma+1}{2}\frac{1-z_1}{z_1}}\,\cot(\beta\eta) \tag{53}$$

where
$$\eta = \frac{1}{(\gamma+1)}\int_{z_1}^{z_2}\frac{dz}{1-z}\sqrt{1+\frac{\gamma+1}{2}\frac{1-z}{z}}$$
Or
$$\eta = \frac{1}{\gamma+1}\left[\sqrt{2(\gamma-1)}\left\{\tan^{-1}\sqrt{\frac{\gamma+1}{\gamma-1}\frac{1}{z_1}-1} - \tan^{-1}\sqrt{\frac{\gamma+1}{\gamma-1}\frac{1}{z_2}-1}\right\}\right.$$
$$\left. + \log\left\{\frac{\sqrt{\frac{\gamma+1}{\gamma-1}\frac{1}{z_1}-1}-\sqrt{\frac{2}{\gamma-1}}}{\sqrt{\frac{\gamma+1}{\gamma-1}\frac{1}{z_1}-1}+\sqrt{\frac{2}{\gamma-1}}}\;\frac{\sqrt{\frac{2}{\gamma-1}}+\sqrt{\frac{\gamma+1}{\gamma-1}\frac{1}{z_2}-1}}{\sqrt{\frac{2}{\gamma-1}}-\sqrt{\frac{\gamma+1}{\gamma-1}\frac{1}{z_2}-1}}\right\}\right] \tag{54}$$

Appendix

Interaction of the Jet with the Surrounding Air

The equation for the jet requires

$$\frac{1}{r}\frac{d}{dr}(rh) = \frac{\omega}{u_2} g(r)$$

$g(r)$ actually is a constant g specified by $G(z_2)$ at the exit. Thus

$$h(r) = \frac{\omega}{u_2} g r \qquad (55)$$

Therefore if R is the radius of the jet, then at the boundary of the jet

$$\left.\begin{aligned} \frac{v'}{u_2} &= \beta\left(\frac{R}{z_2}\right) g \\ \frac{p'}{p_2} &= 0 \end{aligned}\right\} \text{ at } r = R \qquad (56)$$

Outside of the jet, the condition of the surrounding air is denoted by the subscript 0. The disturbed motion there is potential with the potential $\varphi(x, r, t)$. The differential equation is

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} = \frac{1}{a_0^2}\frac{\partial^2\varphi}{\partial t^2} \qquad (57)$$

In view of Eqs (6) to (8),

$$\varphi(x, r, t) = W(r)\cos\omega\left(t - \frac{x}{u_2} + \alpha\right) \qquad (58)$$

Then the equation for W is

$$\frac{d^2W}{dr^2} + \frac{1}{r}\frac{dW}{dr} + \left[\left(\frac{\omega}{a_0}\right)^2 - \left(\frac{\omega}{u_2}\right)^2\right] W = 0$$

Thus

$$W = C\, J_0\left(\sqrt{\left(\frac{\omega}{a_0}\right)^2 - \left(\frac{\omega}{u_2}\right)^2}\; r\right) + D\, Y_0\left(\sqrt{\left(\frac{\omega}{a_0}\right)^2 - \left(\frac{\omega}{u_2}\right)^2}\; r\right) \qquad (59)$$

where C and D are constants to be determined by the conditions expressed in Eq (5b). Therefore although there is no pressure oscillation in the jet, the surrounding air does have pressure oscillations at all r, except $r = h$.

3.3

Servo – Stabilization of Combustion in Rocket Motors
火箭发动机燃烧室的伺服稳定

这是作者发表于 1952 年的“Servo – Stabilization of Combustion in Rocket Motors”（火箭发动机燃烧室的伺服稳定）一文的原始演算稿，共有 18 页。

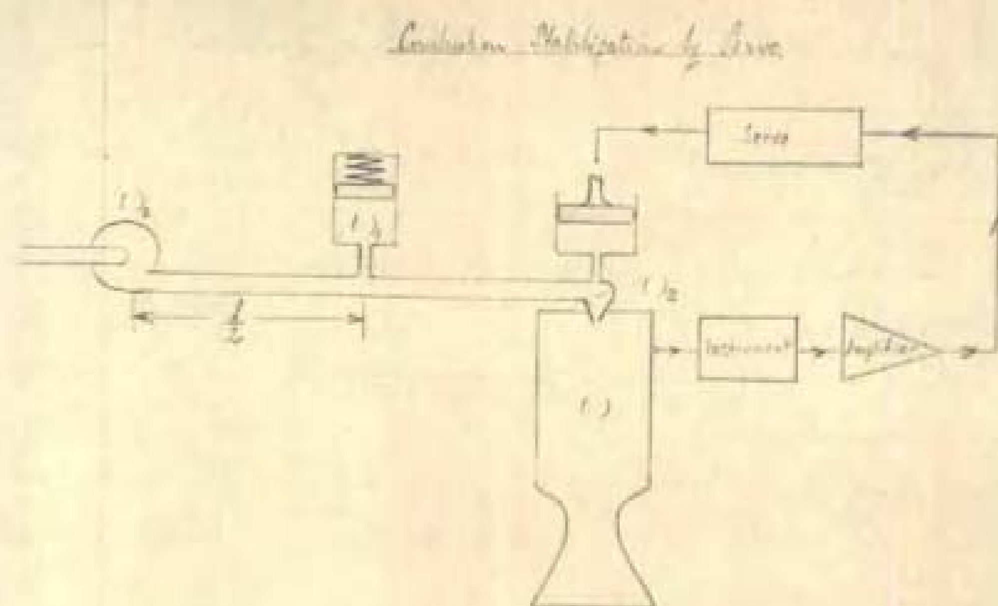
当时已有不少学者，对使用液体推进剂的火箭发动机中的不稳定燃烧现象作出了理论解释，把这一现象看成是，由推进剂的馈送机构和燃烧室所组成的耦合系统失去了稳定性。这里的关键因素在于，从注入推进剂的瞬刻到推进剂燃烧的瞬刻之间，存在着一个时间滞后（简称时滞）。L. Crocco (1951) 曾经分析过这类具有时滞的燃烧稳定性问题，在分析中他假设时滞不受燃烧室压力的影响。

作者想到，为了使火箭发动机的燃烧稳定进行，可以采用反馈控制的方法，即：在不改变推进剂的馈送机构和发动机本身设计的前提下，可以在燃烧室喷嘴前面设置一个由伺服机构控制的容器。当测压仪器测量到了燃烧室压力以后，把测量结果通过一个放大器，变成为伺服机构的输入信号，从而反馈控制推进剂喷入燃烧室的供给速率。考虑到设计时并不掌握有关时滞的确切知识，必须设法做到使系统无条件稳定。也就是说，对于任何一个时滞值，系统都应当是稳定的。为此，作者做了这样一个有时滞的燃烧系统的稳定性分析的研究。

作者采用了萨奇图（Satche diagram）的方法，来讨论时滞可随燃烧室压力变化的更为一般的情况。研究的结果说明伺服稳定方法确实有效。

2

Combustion Stabilization by Servo



$$\frac{\dot m_1 - \bar{\dot m}}{\bar{\dot m}} = -n\,\frac{p_c - \bar p_c}{\bar p_c} \qquad (1)$$

$$\dot m_0 - \dot m_1 = \frac{dC_1}{dt} = \frac{V}{g}\frac{dp_1}{dt}$$

$$p_0 - p_1 = \frac{l}{2A}\frac{d\dot m_0}{dt}$$

$$p_1 - p_2 = \frac{l}{2A}\frac{d\dot m_1}{dt} \qquad \bar p_2 - \bar p = \frac{1}{2}\frac{\bar{\dot m}^2}{\rho A_2^2} = \Delta\bar p$$

$$\dot m_1 - \dot m_2 = \frac{dC_2}{dt}$$

$$p_2 - p = \frac{1}{2}\frac{\dot m_2^2}{\rho A_2^2}$$

$$p_0 - p = \frac{l}{2A}\left[\frac{d\dot m_0}{dt} + \frac{d\dot m_1}{dt}\right] + \frac{1}{2}\frac{\dot m_2^2}{\rho A_2^2}$$

$$(p_0 - \bar p_0) - (p - \bar p) = \frac{l}{2A}\left[\frac{d\dot m_0}{dt} + \frac{d\dot m_1}{dt}\right] + 2(\Delta\bar p)\mu$$

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$$-\frac{\bar{p}_2}{\bar{p}}\frac{1}{x}\left(\frac{\dot{m}_2-\bar{m}}{\bar{m}}\right)-\varphi=2\left(\frac{\Delta\bar{p}}{\bar{p}}\right)\mu+\frac{1}{2A\bar{p}}\left[\frac{d\dot{m}_2}{dt}+\frac{d\dot{m}_2}{dt}\right]$$

$$\dot{m}_3=\dot{m}_1+\chi\frac{dp_1}{dt}=\dot{m}_2+\frac{dC_2}{dt}+\chi\left[\frac{d(p_1-p_2)}{dt}+\frac{dp_2}{dt}\right]$$

$$p_1-p=\frac{1}{2A}\frac{d\dot{m}_2}{dt}+\frac{1}{2}\frac{\dot{m}_2^2}{\rho A_2^2}$$

$$=\frac{1}{2A}\frac{d}{dt}\left[\dot{m}_2+\frac{dC_2}{dt}\right]+\frac{1}{2}\frac{\dot{m}_2^2}{\rho A_2^2}$$

$$\frac{d(p_1-p_2)}{dt}\cong\frac{1}{2A}\frac{d^2}{dt^2}\left[\dot{m}_2+\frac{dC_2}{dt}\right]+\frac{\bar{m}}{\rho A_2^2}\frac{d\dot{m}_2}{dt}$$

$$=\frac{1}{2A}\frac{d^2}{dt^2}\left[\bar{m}\mu+\frac{dC_2}{dt}\right]+2(\Delta\bar{p})\frac{d\mu}{dt}$$

$$\frac{dp_2}{dt}=\bar{p}\frac{d\mu}{dt}$$

$$\dot{m}_3/\bar{m}=(\mu+1)+\frac{dx_2}{dz}+\chi\left[\frac{1}{2A}\frac{1}{\bar{p}_0}\frac{d^2}{dz^2}\left(\mu+\frac{dx_2}{dz}\right)+2\frac{(\Delta\bar{p})}{\bar{m}}\frac{1}{\bar{p}_0}\frac{d\mu}{dz}+\frac{\bar{p}}{\bar{m}\bar{p}_0}\frac{d\mu}{dz}\right]$$

$$=(\mu+1)+\frac{dx_2}{dz}+2\frac{(\Delta\bar{p})}{\bar{m}\bar{p}_0}\chi\left[\frac{J}{2}\frac{d^2}{dz^2}\left(\mu+\frac{dx_2}{dz}\right)+\frac{d\mu}{dz}+P\frac{d\mu}{dz}\right]$$

Put $\quad 2\frac{\Delta\bar{p}}{\bar{m}\bar{p}_0}\chi=E$

$$\left(\frac{\dot{m}_3}{\bar{m}}-1\right)=\mu+\frac{dx_2}{dz}+E\left[\frac{J}{2}\frac{d^2}{dz^2}\left(\mu+\frac{dx_2}{dz}\right)+\frac{d\mu}{dz}+P\frac{d\mu}{dz}\right]$$

$$\dot{m}_3=\dot{m}_2+\frac{dC_2}{dt}$$

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$$-\left(\frac{\Delta\tau}{p}+1\right)\frac{1}{\alpha}\left[\mu+\frac{dx_2}{dt}+E\left\{\frac{J}{2}\frac{d^2}{dt^2}\left(\mu+\frac{dx_2}{dt}\right)+\frac{d\mu}{dt}+p\frac{d\varphi}{dt}\right\}\right]-\varphi$$

$$=2\left(\frac{\Delta\tau}{p}\right)\mu+\frac{\Delta\tau}{p}J\left[\frac{d\mu}{dt}+\frac{d^2x_2}{dt^2}+E\left\{\frac{J}{2}\frac{d^3}{dt^3}\left(\mu+\frac{dx_2}{dt}\right)+\frac{d\mu}{dt}+p\frac{d^2\varphi}{dt^2}\right\}\right.$$
$$\left.+\frac{d\mu}{dt}+\frac{d^2x_2}{dt^2}\right]$$

$$-\frac{p+\frac{1}{2}}{\alpha}\left[\mu'+\frac{dx_2}{dt}+E\left\{\frac{J}{2}\frac{d^2}{dt^2}\left(\mu+\frac{dx_2}{dt}\right)+\frac{d\mu'}{dt}+p\frac{d\varphi'}{dt}\right\}\right]-p\varphi'$$

$$=\mu'+J\left[\frac{d\mu}{dt}+\frac{d^2x_2}{dt^2}+\frac{E}{2}\left\{\frac{J}{2}\frac{d^3}{dt^3}\left(\mu+\frac{dx_2}{dt}\right)+\frac{d\mu'}{dt^2}+p\frac{d\varphi'}{dt^2}\right\}\right]$$

$$p\left\{1+\frac{p+\frac{1}{2}}{\alpha}E\frac{d}{dt}+\frac{JE}{2}\frac{d^2}{dt^2}\right\}\varphi+\left\{\left(1+\frac{p+\frac{1}{2}}{\alpha}\right)+\left(\frac{p+\frac{1}{2}}{\alpha}E+J\right)\frac{d}{dt}+\left(\frac{p+\frac{1}{2}}{\alpha}\frac{EJ}{2}+\frac{EJ}{2}\right)\frac{d^2}{dt^2}\right.$$
$$\left.+\frac{EJ^2}{4}\frac{d^3}{dt^3}\right\}\mu$$
$$+\left\{\frac{p+\frac{1}{2}}{\alpha}\frac{d}{dt}+J\frac{d^2}{dt^2}+\frac{p+\frac{1}{2}}{\alpha}\frac{EJ}{2}\frac{d^3}{dt^3}+\frac{EJ^2}{4}\frac{d^4}{dt^4}\right\}x_2=0$$

$$\begin{vmatrix} p+(1-\pi)+\pi e^{-\beta p} & -e^{-\beta p} & 0 \\ p\left[1+\frac{P\sigma_2}{\alpha}Ep+\frac{JE}{2}p^2\right] & \left(1+\frac{P\sigma_2}{\alpha}\right)+\left(\frac{P\sigma_2}{\alpha}E+J\right)p+\frac{EJ}{2}\left(1+\frac{P\sigma_2}{\alpha}\right)p^2+\frac{EJ^2}{4}p^3 & \left(\frac{P\sigma_2}{\alpha}p+Jp^2+\frac{P\sigma_2}{\alpha}\frac{EJ}{2}p^3+\frac{EJ^2}{4}p^4\right) \\ F(p) & 0 & -1 \end{vmatrix}=0$$

$$\left[p+(1-\pi)+\pi e^{-\beta p}\right]\left[\left(1+\frac{P\sigma_2}{\alpha}\right)+\left(\frac{P\sigma_2}{\alpha}E+J\right)p+\frac{EJ}{2}\left(1+\frac{P\sigma_2}{\alpha}\right)p^2+\frac{EJ^2}{4}p^3\right]$$

$$+e^{-\beta p}p\left[1+\frac{P\sigma_2}{\alpha}Ep+\frac{JE}{2}p^2\right]+pF(p)e^{-\beta p}\left(\frac{P\sigma_2}{\alpha}+Jp+\frac{P\sigma_2}{\alpha}\frac{EJ}{2}p^2+\frac{EJ^2}{4}p^3\right)=0.$$

$$\left[p+(1-\pi)\right]\left[\frac{EJ^2}{4}p^3+\frac{EJ}{2}\left(1+\frac{P\sigma_2}{\alpha}\right)p^2+\left(\frac{P\sigma_2}{\alpha}E+J\right)p+\left(1+\frac{P\sigma_2}{\alpha}\right)\right]$$

$$+e^{-\beta p}\left[\pi\frac{EJ^2}{4}p^3+\left\{\pi\frac{EJ}{2}\left(1+\frac{P\sigma_2}{\alpha}\right)+\frac{PJE}{2}\right\}p^2+\left\{\pi\left(\frac{P\sigma_2}{\alpha}E+J\right)+\frac{P\sigma_2}{\alpha}EP\right\}p+\left\{\pi\left(1+\frac{P\sigma_2}{\alpha}\right)+P\right\}\right.$$

$$\left.+pF(p)\left\{\frac{EJ^2}{4}p^3+\frac{P\sigma_2}{\alpha}\frac{EJ}{2}p^2+Jp+\frac{P\sigma_2}{\alpha}\right\}\right]=0.$$

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$$e^{-pb}\left[1+\frac{pF(p)\left\{\frac{EJ^2}{4}p^3+\frac{P+\frac{1}{2}}{\alpha}\frac{EJ}{2}p^2+Jp+\frac{P+\frac{1}{2}}{\alpha}\right\}}{n\frac{EJ^2}{4}p^3+\frac{EJ}{2}\left\{n\left(1+\frac{P+\frac{1}{2}}{\alpha}\right)+P\right\}p^2+\left\{\left(\frac{P+\frac{1}{2}}{\alpha}\right)E(n+P)+nJ\right\}p+\left\{n\left(1+\frac{P+\frac{1}{2}}{\alpha}\right)+P\right\}}\right]$$

$$+\left[p+(1-n)\right]\frac{\frac{EJ^2}{4}p^3+\frac{EJ}{2}\left(1+\frac{P+\frac{1}{2}}{\alpha}\right)p^2+\left(\frac{P+\frac{1}{2}}{\alpha}E+J\right)p+\left(1+\frac{P+\frac{1}{2}}{\alpha}\right)}{n\frac{EJ^2}{4}p^3+\frac{EJ}{2}\left\{n\left(1+\frac{P+\frac{1}{2}}{\alpha}\right)+P\right\}p^2+\left\{\left(\frac{P+\frac{1}{2}}{\alpha}\right)E(n+P)+nJ\right\}p+\left\{n\left(1+\frac{P+\frac{1}{2}}{\alpha}\right)+P\right\}}=1$$

$$e^{-pb}\left[1-F_1(p)\right]-g_2(p)=n$$

$$F_1(p)=-\frac{pF(p)\left\{\frac{EJ^2}{4}p^3+\frac{P+\frac{1}{2}}{\alpha}\frac{EJ}{2}p^2+Jp+\frac{P+\frac{1}{2}}{\alpha}\right\}}{n\frac{EJ^2}{4}p^3+\frac{EJ}{2}\left\{n\left(1+\frac{P+\frac{1}{2}}{\alpha}\right)+P\right\}p^2+\left\{\left(\frac{P+\frac{1}{2}}{\alpha}\right)(n+P)E+nJ\right\}p+\left\{n\left(1+\frac{P+\frac{1}{2}}{\alpha}\right)+P\right\}}$$

$$g_2(p)=-\left[p+(1-n)\right]\frac{\frac{EJ^2}{4}p^3+\frac{EJ}{2}\left(1+\frac{P+\frac{1}{2}}{\alpha}\right)p^2+\left(\frac{P+\frac{1}{2}}{\alpha}E+J\right)p+\left(1+\frac{P+\frac{1}{2}}{\alpha}\right)}{n\frac{EJ^2}{4}p^3+\frac{EJ}{2}\left\{n\left(1+\frac{P+\frac{1}{2}}{\alpha}\right)+P\right\}p^2+\left\{\left(\frac{P+\frac{1}{2}}{\alpha}\right)(n+P)E+nJ\right\}p+\left\{n\left(1+\frac{P+\frac{1}{2}}{\alpha}\right)+P\right\}}$$

When $p=0$,

$$g_2(0)=-\frac{1-n}{n}\frac{1+\frac{P+\frac{1}{2}}{\alpha}}{1+\frac{P+\frac{1}{2}}{\alpha}+\frac{P}{n}}$$

When $p\to\infty$

$$g_2(\infty)\cong-\left[p+(1-n)\right]\frac{1}{n}\left[1-\frac{2}{J}\frac{P}{n}\frac{1}{p}\cdots\right]$$

$$=-\frac{1}{n}\left[p+(1-n)-\frac{2}{J}\frac{P}{n}\cdots\right]$$

$$=-\left[\frac{p}{n}+\left(\frac{1-n}{n}-\frac{2}{J}\frac{P}{n^2}\right)\right]$$

$$\boxed{\lim_{n\to\infty} n\, f_2(in) = -\left[\frac{1-n}{n} - \frac{2P}{n^2 J}\right]}$$

$$P = \frac{\bar{p}}{2\lambda\bar{p}}, \quad J = \frac{l\bar{x}}{2\lambda\bar{p} A b_f}, \quad E = \frac{2\lambda\bar{p}}{\bar{x} l_f}\frac{dC}{dp}.$$

$$(\sim 3/2) \qquad (\sim 4?) \qquad (0.25)$$

$$n = 1/2.$$

$$C = \rho\pi r^2 l, \quad \frac{dC}{dp} = 2\pi r l \rho \frac{dr}{dp} = 2\pi r^2 l \rho \frac{d\varepsilon}{dp} = 2\pi r^2 l \rho \frac{1}{E}\frac{d\sigma}{dp}$$

$$= 2\pi r^2 l \rho \frac{1}{E}\frac{R}{t}$$

$$E = \frac{2\lambda\bar{p}}{\pi R v b_f} 2\pi R l \rho \frac{1}{E}\frac{R}{t} \qquad \alpha = \begin{cases} \infty \\ 1 \end{cases}$$

$$= 4\frac{\lambda\bar{p}}{E}\frac{R}{t}\frac{l/b_f}{v}$$

$$= 4\frac{100}{10\times10^6}\, 10\, \frac{10}{15\times0.001}$$

$$= \frac{4}{10\times1} = \underline{\underline{0.4}}$$

2

Case 1

$$\kappa = \frac{1}{2}, \quad \rho = 3/2, \quad J = 4, \quad E = 1/4, \quad \alpha = \infty$$

$$g_2(p) = -\frac{(p+\frac{1}{2})(p^3+\frac{1}{2}p^2+4p+1)}{\frac{1}{2}p^3+p^2+2p+2} = -\frac{1}{2}\frac{(2p+1)(2p^3+p^2+8p+2)}{p^3+2p^2+4p+4}$$

$$g_2(i\omega) = -\frac{1}{2}\frac{(1+2i\omega)\{(2-\omega^2)+i\omega(8-2\omega^2)\}}{\{(4-2\omega^2)+i\omega(4-\omega^2)\}}$$

$$= -\frac{1}{2}\frac{\{(2-\omega^2)-2\omega^2(8-2\omega^2)\}+i\omega\{2(2-\omega^2)+(8-2\omega^2)\}}{\{(4-2\omega^2)+i\omega(4-\omega^2)\}}$$

$$= -\frac{1}{2}\frac{\{(2-17\omega^2+4\omega^4)+i\omega(12-4\omega^2)\}\{(4-2\omega^2)-i\omega(4-\omega^2)\}}{(4-2\omega^2)^2+\omega^2(4-\omega^2)^2}$$

$$= -\frac{1}{2}\frac{\{(4-2\omega^2)(2-17\omega^2+4\omega^4)+\omega^2(4-\omega^2)(12-4\omega^2)\}+i\omega\{(4-2\omega^2)(12-4\omega^2)-(4-\omega^2)(2-17\omega^2+4\omega^4)\}}{(4-2\omega^2)^2+\omega^2(4-\omega^2)^2}$$

$$g_2(i\omega) = -\frac{1}{2}\frac{(4-2\omega^2)(2-17\omega^2+4\omega^4)+\omega^2(4-\omega^2)(12-4\omega^2)}{(4-2\omega^2)^2+\omega^2(4-\omega^2)^2} - i\frac{\omega}{2}\frac{(4-2\omega^2)(12-4\omega^2)-(4-\omega^2)(2-17\omega^2+4\omega^4)}{(4-2\omega^2)^2+\omega^2(4-\omega^2)^2}$$

Let
$$F_1(p) = \frac{5/6}{p+1}$$

$$1 - F_1(p) = \frac{p+1/6}{p+1}$$

$$\frac{1}{1-F_1(p)} = \frac{p+1}{p+1/6}$$

$$\frac{1}{1-F_1(i\omega)} = \frac{(1+i\omega)}{1/6+i\omega} = 6\,\frac{1+i\omega}{1+6i\omega} = 6\,\frac{(1+i\omega)(1-6i\omega)}{1+36\omega^2}$$

$$= \frac{6}{1+36\omega^2}\left[(1+6\omega^2)-5i\omega\right]$$

3

Let $F(p) = -\frac{7}{16}\frac{1}{p^2}$

Then $F_1(p) = +\frac{7}{16}\frac{(p^2+4)}{\frac{1}{2}p^3+p^2+2p+2}$ Not good

$$= +\frac{7}{8}\frac{p^2+4}{p^3+2p^2+4p+4}$$

$$1-F_1(p) = \frac{8p^3+9p^2+32p+4}{8(p^3+2p^2+4p+4)}$$

$$g_2^*(p) = g_2(p)/1-F_1(p) = -\frac{1}{2}\frac{(2p+1)(2p^3+p^2+8p+2)}{p^3+2p^2+4p+4}\;\frac{8(p^3+2p^2+4p+4)}{8p^3+9p^2+32p+4}$$

$$= -4\frac{(2p+1)(2p^3+p^2+8p+2)}{8p^3+9p^2+32p+4}$$

$$g_2^*(i\omega) = -4\frac{(1+2i\omega)\{(2-\omega^2)+i\omega(8-2\omega^2)\}}{(4-9\omega^2)+i\omega(32-8\omega^2)}$$

$$= -4\frac{(2-\omega^2)-2\omega^2(8-2\omega^2)+i\omega\{(8-2\omega^2)+2(2-\omega^2)\}}{(4-9\omega^2)+i\omega(32-8\omega^2)}$$

$$= -4\frac{\{(2-17\omega^2+4\omega^4)+i\omega(12-4\omega^2)\}\{(4-9\omega^2)-i\omega(32-8\omega^2)\}}{(4-9\omega^2)^2+\omega^2(32-8\omega^2)^2}$$

$$R\,g_2^*(i\omega) = -\frac{(2-17\omega^2+4\omega^4)(4-9\omega^2)+\omega^2(12-4\omega^2)(32-8\omega^2)}{\frac{1}{4}\left[(4-9\omega^2)^2+\omega^2(32-8\omega^2)^2\right]} \cdots$$

$$I\,g_2^*(i\omega) = -\omega\frac{(12-4\omega^2)(4-9\omega^2)-(32-8\omega^2)(2-17\omega^2+4\omega^4)}{\frac{1}{4}\left[(4-9\omega^2)^2+\omega^2(32-8\omega^2)^2\right]}$$

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ω	ω^2	$4-9\omega^2$	$32-8\omega^2$	$2-17\omega^2+4\omega^4$	$12-6\omega^2$	Denominator	$\mathcal{R}\frac{q}{p}$	$\mathcal{I}\frac{q}{p}$
0	0	4	32	2	12	4	−2.0000	0
0.4	0.16	2.56	30.72	−0.6176	11.36	29.3871	−1.3775	−0.6880
0.8	0.64	−1.76	26.88	−7.2416	9.44	116.2799	−1.5049	−1.2229
1.2	1.44	−8.96	20.48	−14.1856	6.44	171.0653	−1.8188	−1.6458
1.4	1.96	−13.64	16.32	−15.9536	4.16	177.0202	−1.9810	−1.6104
1.6	2.56	−19.04	11.52	−15.2256	1.76	175.5651	−1.9555	−1.3015
1.7	2.89	−22.01	8.88	−13.7216	0.44	178.0823	−1.3593	−1.0707
1.9	3.61	−28.49	3.12	−7.0416	−2.44	211.7053	−0.8447	−0.8267
2.0	4.00	−32.00	0	−2.	−4	256.	−0.2500	−1
2.4	5.76	−47.84	−14.08	36.7904	−11.04	857.6412	+1.0012	
2.8	7.84	−66.56	−30.72	114.5824	−19.36			
3.2	10.24	−88.16	−49.92	247.3504	−28.96			
3.6	12.96	−112.64	−71.68	453.5264	−39.84			
4.0	16.00	−140	−96	754	−52			
4.4	19.36	−170.24	−122.88	1172.1184	−65.44			
4.8	23.04	−203.36	−152.32	1722.6864	−80.16			
5.2	27.04	−239.36	−184.32	2466.9664	−91.16			

Not good

Let us make

$$g_2^*(p) = \frac{g_2(p)}{1-F_1(p)} = -K\frac{(p+p_1)(p+p_2)}{(p+p_3)}$$

For p large, $g_2^*(p) = -K[p + (p_1+p_2-p_3) + \cdots]$

Now $$g_2(p) = -2\frac{(p+\frac{1}{2})(1+\frac{1}{2}\frac{1}{p}\cdots)}{(1+2\frac{1}{p}\cdots)} = -2[p-1\cdots]$$

Thus $K = 2$, $\boxed{p_1+p_2-p_3 = -1}$

Also $\frac{Kp_1p_2}{p_3} = 2$, $\frac{p_1p_2}{p_3} = 1$

$\boxed{p_1p_2 = p_3}$

Let $p_1 = \frac{1}{2}$, then $p_2 = 2p_3$, or $p_3 = -3 \cdot \frac{1}{2} = -;$
No good

Let $p_1 = \frac{3}{2}$, then $\frac{3}{2}p_2 = p_3$

$\frac{5}{2} = \frac{1}{2}p_2$, $p_2 = 5$, $p_3 = \frac{15}{2}$

$$g_2^*(p) = -2\frac{(p+\frac{3}{2})(p+5)}{(p+\frac{15}{2})} = -2\frac{(2p+3)(p+5)}{(2p+15)}$$

~~Let $p_1 = \frac{4}{3}$, $\frac{4}{3}p_2 = p_3$, $\frac{7}{3} = \frac{1}{3}p_2$~~

Let $p_1 = 2$, $2p_2 = p_3$, $p_2 = 3$, $p_3 = 6$

$$\boxed{g_2^*(p) = -2\frac{(p+2)(p+3)}{(p+6)}}$$

12

Then $$1-F_1(p)=\frac{f_3(p)}{p^4 f_2(p)}=\frac{1}{4}\,\frac{(2p+1)(2p^3+p^2+8p+2)(p+6)}{(p^3+2p^2+4p+4)(p+2)(p+3)}$$

$$F_1(p)=\frac{(p^3+2p^2+4p+4)(p+2)(p+3)-(p+\frac{1}{2})(p+6)(p^3+\frac{1}{2}p^2+4p+1)}{(p^3+2p^2+4p+4)(p+2)(p+3)}$$

$$=\frac{\frac{39}{4}p^3+\frac{15}{2}p^2+\frac{51}{2}p+21}{(p+2)(p+3)(p^3+2p^2+4p+4)}$$

$$=-\frac{p^2F(p)(p^2+4)}{\frac{1}{2}p^3+p^2+2p+2}$$

$$=-\frac{2p^2F(p)(p^2+4)}{p^3+2p^2+4p+4}$$

So
$$\boxed{F(p)=-\frac{\frac{39}{4}p^3+\frac{15}{2}p^2+\frac{51}{2}p+21}{2p^2(p^2+4)(p+2)(p+3)}}$$

$$f(p)=\frac{39}{4}p^3+\frac{15}{2}p^2+\frac{51}{2}p+21$$

$$=9.75p^3+7.5p^2+25.5p+21$$

$$f'(p)=29.25p^2+15p+25.5$$

$f(-0.8)=0.408$, $f'(-0.8)=32.22$

$f(-0.8126)=0.0005$

$$f(p)=(p+0.8126)(9.75p^2-0.4228p+25.8936)$$

$$\boxed{F(p)=-4.875\frac{(p+0.8126)(p^2-0.04337p+2.6558)}{p^2(p+2)(p+3)(p^2+4)}}\quad\leftarrow$$

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$$g_2^*(p) = -2\frac{(p+2)(p+3)}{(p+6)}$$

$$g_2^*(i\omega) = -2\frac{(2+i\omega)(3+i\omega)}{(6+i\omega)} = -2\frac{[(6-\omega^2)+5i\omega](6-i\omega)}{(36+\omega^2)}$$

$$= -2\frac{[\{6(6-\omega^2)+5\omega^2\}+i\omega\{30-(6-\omega^2)\}]}{(36+\omega^2)}$$

$$= -2\frac{36-\omega^2}{36+\omega^2} - 2i\omega\frac{24+\omega^2}{36+\omega^2}$$

$$\Re\, g_2^*(i\omega) = -2\frac{36-\omega^2}{36+\omega^2}\;;\quad \Im\, g_2^*(i\omega) = -2\omega\frac{24+\omega^2}{36+\omega^2}$$

ω
0
0.4
0.8
1.2
1.4
1.6
1.7
1.9
2.0
2.4
2.8
3.2
3.6
4.0

14

ω	ω^2	$36+\omega^2$	$\mathfrak{R}\, g_2^*(i\omega)$	$\mathfrak{I}\, g_2^*(i\omega)$
0	0	36	-2.0000	0
0.4	0.16	36.16	-1.9823	-0.3345
0.8	0.64	36.64	-1.9301	-1.0760
1.2	1.44	37.44	-1.8462	-1.6308
1.4	1.96	37.96	-1.7935	-1.9149
1.6	2.56	38.56	-1.7344	-2.2041
1.7	2.89	38.89	-1.7028	-2.3509
1.9	3.61	39.61	-1.6354	-2.6488
2.0	4.00	40	-1.6000	-2.8000
2.4	5.76	41.76	-1.4483	-3.4207
2.8	7.84	43.84	-1.2847	-4.0872
3.2	10.24	46.24	-1.1142	-4.7391
3.6	12.96	48.96	-0.9412	-5.4353
4.0	16.00	52	-0.7692	-6.1538
4.4	19.36	55.36	-0.6011	-6.8925
4.8	23.04	59.04	-0.4390	-7.6488
5.2	27.04	63.04	-0.2843	-8.4103
5.6	31.36	67.36	-0.1378	-9.2048
6.0	36	72	0	-10.

(15)

<u>Case 2</u>

$$\lambda=\frac{1}{2},\quad P=3/2,\quad J=4,\quad E=1/4,\quad \alpha=1$$

$$f_2(p)=-\left(p+\frac{1}{2}\right)\frac{p^3+\frac{3}{2}p^2+\frac{9}{2}p+3}{\frac{1}{2}p^3+\frac{3}{2}p^2+3p+3}$$

$$=-\frac{1}{2}\frac{(2p+1)(2p^3+3p^2+9p+6)}{p^3+3p^2+6p+6}$$

$$f_2(i\omega)=-\frac{1}{2}\frac{(1+2i\omega)\{(6-3\omega^2)+i\omega(9-2\omega^2)\}}{(6-3\omega^2)+i\omega(6-\omega^2)}$$

$$=-\frac{1}{2}\frac{\{(6-3\omega^2)-2\omega^2(9-2\omega^2)\}+i\omega\{(9-2\omega^2)+2(6-3\omega^2)\}}{(6-3\omega^2)+i\omega(6-\omega^2)}$$

$$=-\frac{1}{2}\frac{\{(6-21\omega^2+4\omega^4)+i\omega(21-8\omega^2)\}\{(6-3\omega^2)-i\omega(6-\omega^2)\}}{(6-3\omega^2)^2+\omega^2(6-\omega^2)^2}$$

$$\Re f_2(i\omega)=-\frac{1}{2}\frac{(6-21\omega^2+4\omega^4)(6-3\omega^2)+\omega^2(21-8\omega^2)(6-\omega^2)}{(6-3\omega^2)^2+\omega^2(6-\omega^2)^2}$$

$$\Im f_2(i\omega)=-\frac{1}{2}\omega\frac{(21-8\omega^2)(6-3\omega^2)-(6-21\omega^2+4\omega^4)(6-\omega^2)}{(6-3\omega^2)^2+\omega^2(6-\omega^2)^2}$$

ω	ω^2	$6-3\omega^2$	$6-\omega^2$	$6-26\omega^2+\omega^4$	$21-8\omega^2$	Denominator	$\Re g_2(i\omega)$	$\Im g_2(i\omega)$
0	0	6	6	6	21	72	-0.5000	0
0.4	0.16	5.52	5.84	2.7424	19.72	71.8546	-0.4671	-0.5168
0.8	0.64	4.08	5.36	-5.8016	15.88	70.0666	-0.4396	-1.0948
1.2	1.44	1.68	4.56	-15.9456	9.48	65.5304	-0.5411	-1.6232
1.4	1.96	0.12	4.04	-20.5776	5.32	64.0094	-0.6195	-1.8222
1.6	2.56	-1.68	3.44	-21.5456	0.52	66.2528	-0.3157	-1.7694
1.7	2.89	-2.67	3.11	-21.2024	-2.12	70.1626	-0.5400	-1.7424
1.9	3.61	-4.83	2.39	-17.6816	-7.88	87.7992	-0.1781	-1.7343
2.0	4	-6	2	-14	-11	104	+0.0385	-1.8077
2.2	4.84	-8.52	1.16	-1.9776	-17.72	158.2062	0.5245	-2.1307
2.4	5.76	-11.28	0.24	+17.7504	-25.08	255.1404	0.9207	-2.6211
2.8	7.84	-17.52	-1.84	87.2204	-41.72	666.9870	1.3888	-2.7462
3.2	13.24	-24.72	-4.24	218.5504	-60.92	1590.3360	1.6071	-4.8251
3.6	12.96	-32.88	-6.96	445.6864	-82.68	3099.7548	1.7299	-5.8685
4.0	16	-42	-10	794	-107	6728	1.7778	-6.7999
4.4	19.36	-52.08	-13.36	1098.6784	-133.88	12225.7762	1.8214	-7.7025
4.8	23.04	-63.12	-17.04	1865.5264	-163.32	21348.1514	1.8618	-8.6264
5.2	27.04	-75.12	-21.04	2262.8464	-195.32	35026.3478	1.8942	-9.5045

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$$g_2^*(p) = -2\,\frac{(p+2)(p+3)}{(p+6)}$$

$$1 - F_1(p) = \frac{1}{4}\,\frac{(2p+1)(2p^3+3p^2+9p+6)}{(p^3+3p^2+6p+6)}\,\frac{(p+6)}{(p+2)(p+3)}$$

$$= \frac{(p+\frac{1}{2})(p+6)(p^3+\frac{3}{2}p^2+\frac{9}{2}p+3)}{(p+2)(p+3)(p^3+3p^2+6p+6)}$$

$$(p+2)(p+3)(p^3+3p^2+6p+6) = (p^2+5p+6)(p^3+3p^2+6p+6)$$

$$\begin{aligned} = p^5 + 3p^4 + 6p^3 + 6p^2 & \\ + 5p^4 + 15p^3 + 30p^2 + 30p & \\ 6p^3 + 18p^2 + 36p + 36 & \\ \hline p^5 + 8p^4 + 27p^3 + 54p^2 + 66p + 36 & \end{aligned}$$

$$(p+\tfrac{1}{2})(p+6)(p^3+\tfrac{3}{2}p^2+\tfrac{9}{2}p+3) = (p^2+\tfrac{13}{2}p+3)(p^3+\tfrac{3}{2}p^2+\tfrac{9}{2}p+3)$$

$$\begin{aligned} = p^5 + \tfrac{3}{2}p^4 + \tfrac{9}{2}p^3 + 3p^2 & \\ + \tfrac{13}{2}p^4 + \tfrac{39}{4}p^3 + \tfrac{117}{4}p^2 + \tfrac{39}{2}p & \\ 3p^3 + \tfrac{9}{2}p^2 + \tfrac{27}{2}p + 9 & \\ \hline p^5 + 8p^4 + (4+9+3+\tfrac{1}{2}+\tfrac{3}{4})p^3 + (3+29+\tfrac{1}{4}+4+\tfrac{1}{2})p^2 & \\ (17+\tfrac{1}{4}) \qquad + (19+\tfrac{1}{2}+13+\tfrac{1}{2})p + 9 & \\ 33 & \end{aligned}$$

[illegible]

$$F_1(p) = \frac{9\frac{3}{4}p^3 + 17\frac{1}{4}p^2 + 33p + 27}{(p+2)(p+3)(p^3+3p^2+6p+6)}$$

$$= -\frac{pF(p)\{p^3+p^2+4p+2\}}{\frac{1}{2}p^3+\frac{3}{2}p^2+3p+3} = -\frac{2pF(p)(p^3+p^2+4p+2)}{p^3+3p^2+6p+6}$$

$$\boxed{F(p) = -\frac{9.75p^3+17.25p^2+33p+27}{2p(p^3+p^2+4p+2)(p+2)(p+3)}}$$

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$f_1(p) = p^3 + p^2 + 4p + 2 \qquad f_1(-0.55) = -0.0639$

$f_1'(p) = 3p^2 + 2p + 4 \qquad f_1'(-0.55) = 3.8075$

$f_1(-0.5332) = 0$

$$f_1(p) = (p + 0.5332)(p^2 + 0.4668p + 3.7511)$$

$f_2(p) = 9.75p^3 + 17.75p^2 + 33p + 27$

$f_2'(p) = 29.25p^2 + 34.5p + 33$

$f_2(-1.0) = 1.5, \quad f_2'(-1) = 27.75$

$f_2(-1.0537) = -0.0264, \quad f_2'(-1.0537) = 29.12$

$f_2(-1.0528) = 0.$

$$f_2(p) = (p + 1.0528)(9.75p^2 + 6.9852p + 25.6460)$$

$$= 9.75(p + 1.0528)(p^2 + 0.7164p + 2.6304)$$

$$\boxed{F(p) = -4.875\,\frac{(p + 1.0528)(p^2 + 0.7164p + 2.6304)}{p(p+2)(p+3)(p + 0.5332)(p^2 + 0.4668p + 3.7511)}}$$

3.4
Engineering Cybernetics
工程控制论

这是作者在 1953 年底在美国加州理工学院开设“工程控制论”一课的讲义基础上所做的修改稿的一部分，在这一修改的基础上形成了 1954 年出版的《Engineering Cybernetics》（工程控制论）一书。这里选印了修改稿中的 13 页，反映作者一丝不苟的精神。

1949 年 N. Wiener（维纳）发表了《控制论》一书，其英文书名是《Cybernetics or Control and Communication in the Animal and the Machine》，开创了控制论这样一门新的学科。从 Wiener 所起的书名便可以看出，控制论是关于既是机器中又是动物中的控制和通讯理论的一门科学，研究的主要问题是一个系统的各个不同部分之间的相互作用的定性性质以及整个系统的运动状态。

基于作者在火箭技术方面的丰富经验，他迅速认识到 Wiener 所创控制论的重要性，很快便运用控制论的原理解决了一批喷气技术中的问题，诸如：火箭喷管的传递函数、远程火箭的自动导航以及火箭发动机燃烧的伺服稳定等问题。作者意识到，不仅在火箭技术的领域内，而且在整个工程技术的范围内，几乎到处存在着被控制的系统或被操纵的系统；而且事实上有关的系统控制的技术已经有了多方面的发展，因此很有必要用一种统观全局的方法，来充分了解和发挥上述导航技术和控制技术等新技术的潜在力量，以更广阔的眼界用更系统的方法来观察有关的问题，不仅可以得到解决旧问题的更有效的新方法，并且可以揭示新的以前没有看到过的前景。于是，作者提出了一门新的技术科学—— 工程控制论。作者首先在 1953 年底在美国加州理工学院开设了“工程控制论”一课，接着于 1954 年出版了《Engineering Cybernetics》（工程控制论）一书。这是一门技术科学，它和控制论的不同之处在于，工程控制论旨在讨论和研究，在工程中实现自动控制与自动调节的理论以及自动控制与调节系统的结构原理。该书的出版在世界科技界引起广泛注意，当即被译成多种文字发行。有趣的是，俄文版的发行，还为平息原苏联对《控制论》创始人 N. Wiener 的批判起到了积极的作用。

Chapter VIII

Linear Systems with Time Lag

In this chapter we shall introduce another new element into our linear systems with constant coefficient: the time lag. By time lag, we mean that the relation between the different variables of the system cannot be expressed as a relation of these variables all taken at some time instant ; but on the contrary, the relation involves variables, some taken at the time instant , and some taken at an earlier instant . Those taken at the instant then lag by the interval behind the variables taken at the instant . This time lag is thus quite different from the characteristic time constant of a first order linear system introduced in Section 3.1. Time lag systems are represented by differential-difference equations of constant coefficients and are more complex than the linear systems studied previously which are represented by differential equations. Systems with time lag were studied by many investigators; for instance, A. Callander, D. Hartree, and A. Porter* and N. Minorsky.** Our interest here is, however, somewhat more restricted. We wish to know: How can we analysis the performance of a feedback servomechanism if there is a characteristic time lag in the system? We wish, specifically, to modify the method of Nyquist of Section 4.3, to time lag systems.

We shall develop the theory by treating a particular example of such systems; the example of stabilizing the combustion in a rocket motor by feedback control. The problem of combustion instability in rocket motors was treated by many authors, the following analysis of combustion lag time originates from the work of L. Crocco.*** For simplicity of calculation,**** we shall consider only the case of so-called low frequency oscillation in a rocket motor using single liquid propellant.

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* A. Callander, D. Hartree and A. Porter, Phil. Trans. Royal Society of London (A), 235:415-444 (1935).

** N. Minorsky, J. Appl. Mechanics (ASME) (:67-71 (1942).

*** L. Crocco, J. American Rocket Society, 21: 163-178 (1951).

**** The following discussion is based upon a paper in J. American Rocket Society, 22:256-262 (1952).

Chapter IX

Linear Systems with Stationary Random Inputs

In the previous chapters, the inputs to a system are considered to be definitely specified functions of time . However, there are many engineering problems for linear systems with constant coefficients where the inputs cannot be so definitely described. An example of such engineering problem is the problem of the motion and the stresses induced in the structure of an airplane wing in turbulent air stream. Here the input can be considered to be the time varying air-flow pattern. But the airflow pattern cannot be described as a definite function of time, but has to be recognized as a random function of time, specified by certain statistical characteristics. It is then evident, the output of the system, the stresses in this case, must be also a random function and can be described also only in statistical terms. The first objective of this chapter is then to find a convenient method of calculating the statistical properties of the output from the specified statistical properties of the input. This forms an easy extention of the early investigations by P. Langevin of the Brownian motion.

Another example of random input is the so-called noise in control signals. The noise is introduced by the disturbances and the fluctuations beyond the control of the designer. The problem of noise is a problem of much research in connection with communications engineering. There the central question is how to design the system so that the effects of the unavoidable noise can be minimized and the useful information of the signal is not destroyed. We shall discuss this particular problem of noise filtering in Chapter 16. The problem of this chapter is, however, somewhat different: In our problem, the random output is the only output of the system. Our purpose in the design of the system, particularly the design of the feedback servomechanism, is to obtain with a given input an output of the desired statistical characteristics. We shall see that the method of transfer function developed in the previous chapters remain useful in the present task.

9.1 Statistical Description of a Random Function

Let us consider a system which generates a random function
Now to formulate the concept of a statistical description of such a random

1.1 Linear Systems of Constant Coefficients

Let us consider the simplest system - a first order system. That is, the differential equation of the system is a first order linear differential equation of constant coefficients. If the system is assumed to be free and is not subjected to "forcing functions", then the differential equation can be written as

$$\frac{dy}{dt} + ky = 0 \tag{1.1}$$

k may be called the spring constant and is real. When there is no variation of y with respect to time, dy/dt vanishes and then Eq. (1.1) requires $y=0$. Therefore the stationary state or the equilibrium state of the system corresponds to $y=0$.

The solution of Eq. (1.1) is

$$y = y_0 e^{-kt} \tag{1.2}$$

where y_0 is the initial value of y or

$$y(0) = y_0 \tag{1.3}$$

y_0 is thus the initial disturbance of the system from the equilibrium state. The behavior of the system for $t>0$, is illustrated in Fig. 1.1 for both positive k and for negative k. It is seen that for $k>0$, the magnitude of y decreases with time. Then as time increases indefinitely, $y \to 0$. Therefore

Fig. 1.1

for $k>0$, the disturbance of the system will eventually disappear. The system can then be said to be stable. When $k<0$, the disturbed motion of the system increases with time and eventually the disturbance will become very large no matter how small the initial displacement is, and will never return to the equilibrium state once disturbed. Such systems are thus unstable.

For systems of higher order, the differential equation will have higher derivatives. The n-th order system has the differential equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = 0 \qquad (1.4)$$

For a physical system, the coefficients $a_{n-1}, \cdots, a_0$ are real. Then the solution of Eq. (1.4) can be written as

$$y = \sum_{i=1}^{n} y_i^{(0)} e^{\alpha_i t} \sin(\beta_i t + \varphi_i) \qquad (1.5)$$

where α_i , β_i are real and are related to the coefficients $a_{n-1}, \cdots, a_0$ and φ_i are the phase angles. The motion of the system is thus only stable if all α_i's are negative. If one of them is positive, the disturbance will eventually diverge, and the system is thus unstable.

From the above examples it is seen that the crucial question to ask about the behavior of a linear system of constant coefficients is the question of stability. Needless to say, the usual aim of an engineering design is stability. The question of stability can be answered, however, once the coefficients of the differential equation are specified. In case of the simple first order system specified by Eq. (1.1), the only information that matters is the sign of the coefficient k .

2. Linear System with Variable Coefficients

If there is a variable parameter in the system under study, the stationary or the equilibrium state of the system can be changed by changing this parameter. It is natural, then, to expect the coefficients of the linear differential equation describing the system to be also functions of this parameter. For instance, the aerodynamic forces acting on an aircraft are functions of the speed of the aircraft. If the speed of the

Chapter II

Method of Laplace Transform

For linear differential equations with constant coefficients and with time t as the independent variable, the method of Laplace transform is particularly useful in finding the solution. Of course, the problem can be solved by a number of other methods; but the method of Laplace transform appeals specially to the engineering scientists in that it reduces all problems to a uniform basis. The procedure of solution is then standardized and a general approach is possible. The theory and practice of Laplace transform is discussed in many texts.* It is not the purpose of the present chapter to do this. The purpose here is rather to give a summary of results which are useful to our discussion in the subsequent chapters for easy reference. For details and proofs, the reader should consult the texts cited.

2.1 Laplace Transform and Inversion Formula

If $y(t)$ is a function of time variable t defined for $t>0$, then the Laplace transform $Y(s)$ of $y(t)$ is defined as**

$$Y(s) = \int_0^\infty e^{-st} y(t)\, dt \tag{2.1}$$

where s is a complex variable having a positive real part, $R>0$. For other values of s, the function $Y(s)$ is defined by the analytic continuation. The dimension of $Y(s)$ is the dimension of y multiplied by time.

When $Y(s)$ is known, the original function for which $Y(s)$ is the Laplace transform can be obtained in all cases by the Inversion Formula:

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* See for instance, H. S. Carslaw and J. C. Jaeger. "Operational Methods in Applied Mathematics", Oxford,(1941); or R. V. Churchill. "Modern Operational Methods in Engineering". McGraw Hill. (1944). For more complete theory, one should consult G. Doetsch, "Theorie und Anwedung der Laplace-Transformation", J. Springer, Berlin, (1937); or D. V. Widder, "The Laplace Transform", Princeton, (1946).

** We shall use throughout capital alphabet to denote the Laplace transform of quantities denoted by a lower case alphabet.

Therefore the error signal vanishes as $t \to \infty$.

Consider now another example of the input: Let the input be sinusoidal,

$$x(t) = x_{m} e^{i\omega t}$$

where x_{m} is the amplitude and ω is the frequency. Then

$$X(s) = \frac{x_{m}}{s - i\omega} \quad (3.12)$$

The output due to the initial condition is the same as before. The output due to input is given by

$$Y_2(s) = x_{m} \frac{1}{(s-i\omega)(\tau_1 s+1)} = \frac{x_{m}}{1+i\omega\tau_1}\left[-\frac{1}{s+\frac{1}{\tau_1}} + \frac{1}{s-i\omega}\right]$$

Therefore according to our dictionary, the output $y_2(t)$ is

$$y_2(t) = -\frac{x_{m}}{1+i\omega\tau_1} e^{-\frac{t}{\tau_1}} + \frac{x_{m}}{1+i\omega\tau_1} e^{i\omega t}$$

e^{-t/τ_1}

The first term is a pure subsidence and the second term is the steady state output. Thus

$$[y(t)]_{steady} / x(t) = \frac{1}{1+i\omega\tau_1} = F(i\omega)$$

This is in full agreement with our general result given in Eq. (2.16). Since

$$\frac{1}{1+i\omega\tau_1} = \frac{1}{\sqrt{1+\omega^2\tau_1^2}} e^{-i\tan^{-1}(\omega\tau_1)} \quad (3.13)$$

the steady state output can be expressed as

$$[y(t)]_{steady} = \frac{x_{m}}{\sqrt{1+\omega^2\tau_1^2}} e^{i[\omega t - \tan^{-1}(\omega\tau_1)]}$$

Therefore the amplitude of the steady state output is reduced by the factor $1/\sqrt{1+\omega^2\tau_1^2}$ in comparison with the input, and the phase of the output lags behind the input by the amount $\tan^{-1}(\omega\tau_1)$. For low

the aileron deflection δ . The equation for the roll angle φ is thus

$$I\frac{d^2\varphi}{dt^2} + L_\varphi \frac{d\varphi}{dt} = k\delta$$

Now let $p = \frac{d\varphi}{dt}$ be roll speed, then the above equation becomes

$$I\frac{dp}{dt} + L_p p = k\delta$$

If the roll speed is zero at $t=0$, then the transformed equation is

$$(Is + L_p)P(s) = k\Delta(s)$$

The transfer function $F(s)$ is thus

$$\frac{P(s)}{\Delta(s)} = F(s) = \frac{k}{Is+L_p} = \frac{k}{L_p}\frac{1}{1+(\frac{I}{L_p})s} \tag{3.36}$$

The behavior of the system is determined by the transfer function is thus similar to the cantilever spring with dashpot and the simple lag network. Here the characteristic time τ_1 is I/L_p . If the damping is very small, then $\tau_1 \to \infty$ and the behavior of the system becomes that of the simple integrator.

3.4 Second Order Systems

Let us return to the cantilever spring with a dashpot, Fig. 3.1. Only now we attach a mass m to the dashpot end. The mass will introduce an inertia force $m\,d^2y/dt^2$, and the equation of motion is now

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = kx$$

with the initial conditions

$$\left.\begin{aligned} y(0) &= y_0 \\ \left(\frac{dy}{dt}\right)_{t=0} &= y_0^{(1)} \end{aligned}\right\} \tag{3.37}$$

The differential equation of motion can be rewritten in more convenient form by introducing the following parameters:

with $G(s)$ given. The method of Evans determines such roots as functions of the gain K, and is thus called the <u>root-locus method.</u> When this is done, any set of specifications on the roots gives a proper choice of the magnitude of K. This method then goes much beyond the mere satisfaction of criterion a) of Section 2, but actually solves the design problem for all three criteria stated in that section.

Now let $G(s)$ be specified by its zeros $p_1, p_2, \cdots p_m$ and its poles $q_1, q_2, \cdots q_n$. Then from the definition of gain given by Eqs. (3.16), (3.21) and (3.23),

$$G(s) = A\,\frac{(s-p_1)(s-p_2)\cdots\cdots(s-p_m)}{(s-q_1)(s-q_2)\cdots\cdots(s-q_n)} \tag{4.16}$$

where A ~~is a constant real and positive for all physical systems.~~

$$A = \frac{(-q_1)(-q_2)\cdots\cdots(-q_n)}{(-p_1)(-p_2)\cdots\cdots(-p_m)}$$

~~sical sy~~ ~~or and~~

For physical systems, the polynomials in the numerator and the denominator of $G(s)$ has real coefficients. Then the p's are either real or form complex conjugate pairs. Similarly the q's are either real or form complex conjugate pairs. Therefore A is always real. In engineering systems, usually things are so arranged as to make A not only real but also positive. Hereafter then, we shall consider A to be real and positive. Generally the denominator of $G(s)$ is of equal or higher order than that of the numerator, i.e., $n \geqq m$. Let us express each of the factors in Eq. (4.16) in vector form:

$$\left.\begin{aligned} s-p_1 &= P_1 e^{i\varphi_1} \\ s-p_2 &= P_2 e^{i\varphi_2} \\ &\cdots\cdots\cdots \\ &= \quad i\varphi \end{aligned}\right\} \tag{4.17}$$

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Since A is real and positive, ~~Hence~~ we can write Eq. (4.19) as

$$G(s) = \mathcal{R}\, e^{i\theta} \tag{4.20}$$

where

$$\mathcal{R} = A\,(P_1 P_2 \cdots\cdots P_m / Q_1 Q_2 \cdots\cdots Q_n) \tag{4.21}$$

and

$$\theta = (\varphi_1 + \varphi_2 + \cdots\cdots + \varphi_m) - (\theta_1 + \theta_2 + \cdots\cdots + \theta_n) \tag{4.22}$$

Since P's and Q's are magnitudes of vectors defined by Eqs. (4.18) and (4.19), they are positive. Therefore $\mathcal{R}$ is positive. The basic equation for the roots of inverse system transfer function, Eq. (4.15), is thus

$$\frac{1}{K\mathcal{R}}\, e^{-i\theta} = -1$$

Therefore to satisfy this equation, we must have

$$K\mathcal{R} = 1 \tag{4.23}$$

and

$$\theta = \pm \pi \tag{4.24}$$

The method of Evans consists of two steps: The first step is to determine all s that satisfy the appropriate angle condition of Eq. (4.24). Then knowing such root-locus, we can compute $\mathcal{R}$ and hence K, by Eq. (4.23), for each point on the root-locus. Evans has developed a number of useful rules for plotting the root-locus. We shall explain these rules presently.

Rule 1 For $K = 0$, Eq. (4.15) shows that $G(s) \to \infty$. Thus for $K = 0$, the roots of $1/F_s(s)$ are poles of $G(s)$, or the root-locus starts at the

oscillating servomechanisms are less flexible than are oscillating control servomechnisms in which the oscillation is supplied by an independent generator.

An elementary precaution to be observed, in order that the curve, which is constrained to pass through the point -2, shall avoid the neighborhood of the point -1, is that the curve should interest the real axis, at the point -2, perpendicularly. This implies that the vector $1/F(i\omega)$ should be varying slowly in amplitude, and rapidly in angle, at the frequency at which the system oscillates.

6.6 General Oscillating Control Servomechanism

A relay ~~or a limiter~~ is a non-linear device. But by mixing the signal with a sinusoidal oscillation of high frequency and large amplitude, the output is made to be linear with respect to the signal. Thus the essential concept of oscillating control servomechanisms is the linearization of a non-linear system. J. M. Loeb* has shown that this concept is applicable to any non-linear system, and he calls this method the general linearizing process for non-linear control systems. We shall call the resulting servomechanism the general oscillating control servomechanism.

Let us consider a general function $y(x)$ where y is the output and x is the input. If instead of the variable x, we substitute the sum $x+\epsilon$ where ϵ is much smaller than x. Then if the function $y(x)$ is regular, we can expand $y(x+\epsilon)$ into a Taylor series as

$$y(x+\epsilon) = y(x) + \epsilon\left(\frac{dy}{dx}\right)_x + \epsilon^2\frac{1}{2}\left(\frac{d^2y}{dx^2}\right)_x + \cdots \tag{6.15}$$

We now specify the input x as a periodic function of time t with the period T, and ϵ as a constant. Then it is clear that $y(x)$ is also a periodic function of time with the same period T. Same is true for dy/dx and d^2y/dx^2. Periodic functions can be expanded into Fourier series; thus if we neglect powers of ϵ higher than first, we have

$$\begin{aligned} y(x+\epsilon) \cong\ & a_{00} + \sum_{n=1}^{\infty}\left(a_{0n}\cos n\omega t + b_{0n}\sin n\omega t\right) \\ & + \epsilon\left[a_{10} + \sum_{n=1}^{\infty}\left(a_{1n}\cos n\omega t + b_{1n}\sin n\omega t\right)\right] \end{aligned} \tag{6.16}$$

* J. M. Loeb, Annales des Télécommunications, 5:65-71 (1950).

$$F_2^*(s) = \frac{t_0}{2\pi i} \sum_{n=0}^{\infty} e^{-n t_0 s} \int_{\gamma - i\infty}^{\gamma + i\infty} F_2(q) e^{n t_0 q} dq$$

$$= \frac{t_0}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F_2(q)\, dq \sum_{n=0}^{\infty} e^{-n t_0 (s-q)} = \frac{t_0}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{F_2(q)\, dq}{1 - e^{-t_0(s-q)}} \qquad (7.11)$$

We proceed to evaluate the right-hand member of (7.11) by the method of residues.

The integrand has certain poles, the poles of $F_2(s)$ lying to the left of the path of integration, and other poles, which are the roots of the equation $1 - e^{-t_0(s-q)} = 0$, lying to the right of the path of integration. It is easily seen that the integration upward along the line $\gamma - i\infty$ to $\gamma + i\infty$ is equivalent to integration in the clockwise direction along the closed contour formed by that line and the infinite semicircle in the right half-plane. Hence the right-hand member of Eq. (7.11) is $-t_0$ times the sum of the residues of the integrand with respect to the several roots of the equation $1 - e^{-t_0(s-q)} = 0$.

Now the typical root of the equation is $q = s + 2\pi i m/t_0$, where m is an integer, and the residue of the integrand with respect to that pole is $-\frac{1}{t_0} F_2(s + 2\pi i m/t_0)$. Therefore finally

$$F_2^*(s) = \sum_{m=-\infty}^{m=\infty} F_2(s + 2\pi i m/t_0) \quad , \qquad Re\, s > \gamma \qquad (7.12)$$

This formula gives considerable insight into the properties of $F_2^*(s)$ and at times may be useful in making approximate calculations. However, we can easily obtain an exact representation of $F_2^*(s)$ in finite form.

The function $F_2(s)$ can be represented as the sum of a finite number of partial fractions, thus:

$$F_2(s) = \sum_{k=1}^{n} \frac{a_k}{s - s_k} \qquad (7.13)$$

automatic sensing and measuring control system, i.e., an optimalizing system which automatically holds the airplane at the measured optimum operating conditions.

Of course, a skilled human operator controls the performance of a machine on the optimalizing principle: He watches the instrument readings of the inputs and outputs of the machine, and then uses his knowledge and experience to decide in what directions should the controls be adjusted. The adjusted inputs bring new output readings which have to be interpreted by the operator to determine whether the optimum operating condition is reached or exceeded. New adjustments of the control will have to be made. The continuous adjustment of inputs is the sensing process and the reading of the outputs is the feedback. However, manually-controlled optimalizing systems are necessarily slow in response, and for complicated systems human skill, no matter how developed, is not sufficient. Automatic optimalizing control was conceived by C. S. Draper, Y. T. Li and H. Laning, Jr.* Its application to cruise control of airplane was discussed by J. R. Shull.**

15.2 Principles of Optimalizing Control

The heart of an optimalizing control system is the non-linear component which characterizes the optimum operating conditions. For simplicity of discussion, we shall assume that this basic component has a single input and a single output. For the time being, we shall neglect also any time effects and assume that the output is determined only by the instantaneous value of the input. Since there is an optimum operating point, output as a function of input has a maximum at y_0 and x_0, as shown in Fig. 15.1. It is convenient to refer the output and the input to the optimum point and put the physical input as $x + x_0$, and the physical output as $y + y_0$. The optimum point is then the point $x = y = 0$. The purpose of an optimalizing control is then to search out this optimum point and to keep the

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* Y. T. Li, Instruments, 25: 72-77, 190-193, 228, 324-327, 350-352 (1952). C. S. Draper and Y. T. Li, "Principles of Optimalizing Control Systems and an Application to Internal Combustion Engine", ASME Publications (1951).

** J. R. Shull, Trans. I.R.E. (Electronic Computers), Dec. 1952, pp. 47-51.

(18.28)

Eqs. (18.27) and (18.28) have a first term identical with Eq. (18.1). The additional terms come from the imperfect elements and from the statistical distribution of errors.

With any specified , , and , Eqs. (18.26), (18.27), and (18.28) enable us to compute the distribution function of , the fraction of activated output lines of the complete Scheffer stroke system. We can make this somewhat clearer by reverting to the notation of probability distribution functions. Thus for instance Eq. (18.26) is equivalent to

The probability distribution function of , (; , ;), is thus the result of integrating with respect to and of the joint probability of , , and . Thus

(18.29)

We shall now show that under proper conditions we can obtain almost perfect performance of the multiplexed Scheffer stroke system by increasing . Consider a given fiduciary level . Perfect performance means the implication of , or non-activation of output, by , , or the activation of both inputs; the implication of , by either , or .

3.5

Analysis of Peak - Holding Optimalizing Control

保持最高点控制的分析

这是作者发表于1955年的"Analysis of Peak - Holding Optimalizing Control"（保持最高点控制的分析）一文的部分手稿，共有12页。

对于简单的控制系统来说，系统的性质和特征是事先就知道的。但是，对于较为复杂的系统来说，情况就不同了，这是因为：系统在制造过程中常常产生误差，其形状和性能和早先在实验室中测试的结果不会完全相同；系统在运行过程中也会发生不断的变化，例如遭受磨损和疲劳损伤；而且运行时的环境条件有所改变等。在不完全掌握系统性质和特征的情况下，需要让系统自动寻找最优运转点。C. S. Draper和李耀滋(1951)等提出了自寻最优点的方法，即在控制系统中引入连续"理解"和连续测量的环节，据此进行连续的调节反馈。

上述的最优控制方法在理论上存在两个基本问题，一是控制系统的惯性或者其他动力学现象对控制性能产生的影响，另一是消除噪声干扰的影响。作者的这篇文章是要解决第一个问题，分析一类噪声干扰影响极小的控制系统，目的在于使系统的输出始终保持极大值，作者将这类控制称为保持最高点的控制（Peak - Holding Optimalizing Control）。这里作者假设，系统的动力学影响可以相当准确地用具有时间常数的一阶线性系统近似描述。作者详细地分析了控制原理，并且给出了供设计用的曲线图，对于指定的搜索周期、输入和输出部分的时间常数以及选定的临界指示差值等参数值，曲线图给出有关输入策动速度和搜索损失的结果。

这里选印作者的原始底稿中的12页，其中前8页是有关引言、优化控制的运作原理和问题的数学表述等内容，接着的两页是有关"设计曲线图"部分，而最后两页是反映最重要结果的两张精美制图，即关于输入策动速度N和搜索损失D的设计曲线。

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Analysis of Peak-Holding Optimalizing Control

H. S. Tsien and S. Serdengecti

Introduction

Optimalizing control was invented by C. S. Draper, Y. T. Li and H. Laning, Jr. (Ref. 1 and 2). Their basic idea seems to be as follows: In almost all engineering systems, within the restrictions of operation, there is always an optimum condition of performance. For instance, an internal combustion engine, within the restriction of producing the load torque at the specified speed, there are optimum settings for the manifold pressure and ignition timing for minimum fuel consumption. Another example is an airplane under cruising condition; there under the restriction of engine cruising power, there is an optimum trim setting for maximum cruising speed and assigned attitude. But more important than the existence of an optimum operating point is the fact that the optimum operating point cannot be exactly known in advance because of the natural unpredictable changes in the environment of the engineering system: In the case of internal combustion engine, it is the changes in the temperature and the humidity of the air; in the case of airplane, it is unavoidable changes in the aerodynamic properties of the airplane and the engine performance with age. Therefore if the problem is to operate always near the optimum point in spite of the "drift" of the system, then the control device for the engineering system must be so designed as to search out automatically the optimum point of operation and to confine the operation close to this point. This is the basic idea of optimalizing control.

The possibility of applying Draper's optimalizing control to the [illegible] cruise control of airplanes was discussed by J. R. Shull (Ref. 3)

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Shull emphasized the expected elimination of extensive flight testing of new airplanes for performance determination. Because the optimalizing control will automatically measure the performance whenever the airplane is flown. This in itself would constitute a great saving. But moreover, in critical circumstances such as fly through icing atmosphere, the ability of the optimalizing control to extract the best performance of a radically changed system through ice deposition on the airplane may be of utmost importance.

There are two fundamental problems in the theory of optimalizing control. One of the problem is the dynamic effects of the controlled system on the performance of the control. The other problem is the elimination of the interference by noise. The two problems are somewhat interrelated, because if large deviations from the optimum operating point and hence large loss can be tolerated, then the noise interference will not be critical. The basic design aim of optimalizing control is to have the smallest loss or to operate as close to the optimum point as possible without the danger of having the control misled by the noise interference. Both of these problems were considered by the original inventors of optimalizing control. The noise problem is essentially the problem of detection of a sinusoidal variation with heavy random interference, a subject of much current investigations. The purpose of the present paper is to solve the first problem completely under the assumption that the dynamic properties of the controlled system can be approximated by a first order linear system. We shall begin with a brief review of the operating principles of optimalizing control of the peak-holding type — a type least affected by the noise interference.

(cont'd, see next page)

Principle of Operation

The heart of an optimalizing control system is the nonlinear component which characterizes the optimum operating condition of the controlled system. For simplicity of discussion, it is assumed that this basic component has a single input and a single output. For the time being, the dynamic effects will be neglected and the output is assumed to be determined by the instantaneous value of the input. Since there is an optimum point, output as a function of input has a maximum at the output y_0 at the input x_0, as shown in Fig. 1. It is convenient to refer the output and the input to the optimum point and put the physical input as $x+x_0$, and the physical output as y^*+y_0. The optimum point is then the point $x=y^*=0$. The purpose of an optimalizing control is then to search out this optimum point and to keep the system in the immediate neighborhood of this point. In this neighborhood, the relation between x and y^* can be represented as

$$y^* = -kx^2 \qquad (1)$$

where k is a characteristic constant of the controlled system.

The operation of a peak-holding optimalizing control, neglecting the dynamic effects, then would be as follows: Say the input x is below the optimum value, and is thus negative. The input drive is then set to increase the input at a constant rate. At the time instant 1 (Fig. 2) the input changes from negative to positive and passes through the optimum point. The output y^* is thus maximum at the time instant 1, and is decreasing after the instant 1. Now if an output sensing instrument is so designed as to follow the output exactly when the output is increasing, but hold to the maximum value after the maximum is passed and the output starts to decrease. Then there

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will be a difference between the reading of this output sensing instrument and the output after the time instant 1. This difference is shown in the lower graph of Fig 2. When this difference is built up to a critical value c at the time instant 2, the input drive is tripped and the direction of the input drive is reversed, but still at the same constant rate as before. After the instant 2, the input decreases and the output increases till a maximum in output is again reached at the time instant 3. At time instant 3, the input of course again passes from positive to negative, and the indicated difference between the output sensing instrument and the output itself again builds up. At the time instant 4, the difference reaches the critical value c again and the input drive direction is again reversed. At the time instant 5, the input becomes zero again and another maximum of the output y^* is reached. The period of input variation is thus the time interval from the instant 1 to the instant 5, and consists of a series of straightline segments. The period of output variation is the time interval from the instant 1 to the instant 3, and consists of a series of parabolic arcs. The periodic variations of input and output are called the hunting of the system and the period of output variation is called the hunting period T. The period of input variation is thus $2T$.

The extreme variation of output Δ (Fig. 2) is called the hunting zone. If a is the amplitude of the triangular waves of the input (Fig. 2), then due to Eq. (1),

$$\Delta = ka^2 \tag{2}$$

The difference between the maximum output and the average output of the hunting system is called the hunting loss D (Fig. 2). Because of the fact that the output is a series of parabolic arcs,

$$D = \frac{1}{3}\Delta = \frac{1}{3}ka^2 \qquad (3)$$

In this idealized case, ~~it is evident that~~ the critical indicated difference c between the output sensing instrument and the output itself is equal to Δ, the hunting zone. It is then clear from this discussion that in order to reduce the hunting loss for better efficiency of the system, one must try to reduce the hunting zone or the amplitude of input variation. Unfortunately the critical indicated difference c is also reduced by such modification and a limit is set of the noise interference on the proper tripping operation of the input drive.

The dynamic effects are so far neglected. But in any physical system, this is not possible due to the ever present inertial and damping forces. The output y^* given by Eq. (1) has to be considered then as the fictitious "potential output" but not the actual output y measured by the output indicating and sensing instrument. y^* is equal to y only when the period T of hunting becomes very long. The relation between y^* and y is determined by the dynamical effects. For the conventional engineering system, these dynamical effects is determined by a linear relation. For instance, in case of internal combustion engine, the potential output is essentially the indicated effective pressure generated in the engine cylinders, while the actual output is the brake mean effective pressure of the engine. The dynamical effects are here mainly due to the inertia of the piston, the crankshaft and other moving parts of the engine. For small changes in the operating conditions of engine, such dynamical effects can be represented as a linear differential equation with constant coefficients. Since the reference level of input and output is taken to be the optimum input x_0 and the optimum output y_0, the physical potential output is $y^* + y_0$ and the physical actual output is $y + y_0$. Thus the relation between the

physical potential output and the physical actual output can be written as an operator equation

$$(y+y_0) = F_o\left(\frac{d}{dt}\right)(y^* + y_0) \quad (4)$$

where F_o is generally the quotient of two polynomials in the (time differential) operator $\frac{d}{dt}$. In the language of Laplace transform then, $F_o(s)$ is the transfer function. Let the linear system which transforms the potential output to actual output, the output linear group. Then $F_o(s)$ is, specifically, the transfer function of the output linear group. By implication however, when the dynamic effects are negligible or when $s=0$, the potential output is equal to the actual output. Therefore

$$F_o(0) = 1 \quad (5)$$

Since y_0 in Eq.(4) is a constant independent of time, the condition of Eq.(5) then simplifies that equation to

$$y = F_o\left(\frac{d}{dt}\right) y^* \quad (6)$$

In a similar manner, let x^* be the "potential input" which is actually the forcing function generated by the optimalizing control system but not the actual input x. It is x^* which has the saw-tooth form shown in Fig. 2, but not x. The relation between x and x^* is determined by the inertial and dynamical effects of the input drive system. This input drive system can be called the "input linear group" of the optimalizing control. The operator equation between the potential input x^* and the actual input x is

$$x = F_i\left(\frac{d}{dt}\right) x^* \quad (7)$$

$F_i(s)$ is thus the transfer function of the input linear group. Similar to Eq.(5), the meaning of potential and actual inputs implies

$$F_i(0) = 1 \quad (8)$$

Thus a simple representative block diagram of the complete optimalizing control system can be drawn as shown in Fig 3. The nonlinear components of the system are thus the optimalizing input drive and the controlled system itself.

Formulation of the Mathematical Problem

The general relation between the input x and the output y is determined by the system of Eqs. (1), (6) and (7), with the potential input x^* specified as a saw-tooth curve with period $2T$ and amplitude a. Let ω_0 be the frequency defined by

$$\omega_0 = \frac{2\pi}{T} \tag{9}$$

Then x^* can be expanded into a Fourier series,

$$x^* = \frac{8a}{\pi^2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin (2n+1)\frac{\omega_0 t}{2}$$

$$= \frac{8a}{\pi^2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}\frac{1}{2i}\left[e^{\frac{2n+1}{2}i\omega_0 t} - e^{-\frac{2n+1}{2}i\omega_0 t}\right] \tag{10}$$

Therefore by using Eq (7), the actual input x is given by

$$x = \frac{8a}{\pi^2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}\left[F_i\left(\frac{2n+1}{2}i\omega_0\right)e^{\frac{2n+1}{2}i\omega_0 t} - F_i\left(-\frac{2n+1}{2}i\omega_0\right)e^{-\frac{2n+1}{2}i\omega_0 t}\right] \tag{11}$$

Eqs (1) and (6) then give the actual output as

$$y = \frac{16a^2k}{\pi^4}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+m}}{(2n+1)^2(2m+1)^2}\Big[F_o((n+m+1)i\omega_0)F_i\left(\frac{2n+1}{2}i\omega_0\right)F_i\left(\frac{2m+1}{2}i\omega_0\right)e^{(n+m+1)i\omega_0 t}$$

$$- F_o((n-m)i\omega_0)F_i\left(\frac{2n+1}{2}i\omega_0\right)F_i\left(-\frac{2m+1}{2}i\omega_0\right)e^{(n-m)i\omega_0 t} - F_o((n-m)i\omega_0)F_i\left(-\frac{2n+1}{2}i\omega_0\right)F_i\left(\frac{2m+1}{2}i\omega_0\right)e^{\cdots}$$

$$+ F_o(-(n+m+1)i\omega_0)F_i\left(-\frac{2n+1}{2}i\omega_0\right)F_i\left(-\frac{2m+1}{2}i\omega_0\right)e^{-(n+m+1)i\omega_0 t}\Big] \tag{12}$$

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By comparing Eqs. (11) and (12), it is seen that the input has half the frequency of the output. This is, of course, to be expected from the basic parabolic relation of input and output.

The average of the actual output y with respect to time t, being here referred to the optimum output y_0, gives directly the hunting loss D. Eq. (12) shows that this average value is the sum of terms with $n = m$ from the second and the third terms of that equation. Therefore, using Eq. (5),

$$D = \frac{32 a^2 k}{\pi^4} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} F_i\left(\frac{2n+1}{2} i\omega_0\right) F_i\left(-\frac{2n+1}{2} i\omega_0\right) \qquad (13)$$

This equation can be easily checked by observing that when the dynamic effects are absent, $F_i \equiv 1$, then the series can be easily summed and $D = \frac{1}{3} a^2 k$ as required by Eq. (3). Eq. (13) also shows that the average output and hence the hunting loss are independent of the output linear group. This agrees with the one's physical understanding. Only detailed time variation of the output is modified by the dynamics of the output linear group. In case of an internal combustion engine, the average output specifies the power of the engine. The dynamics of the output linear group is determined by the inertia of the moving parts. The power of the engine is certainly independent of the inertia of the moving parts.

Eqs. (11) to (13) fully determine the performance the optimalizing control system once the values of a, k, ω_0 are specified and the transfer functions $F_i(s)$ and $F_o(s)$ of the input linear group and the output linear group are given. The following sections give the detailed calculations and results for the case first order input and output groups.

<u>First Order Input and Output Groups</u>

The frequency ω_0 of the optimalizing control is usually quite low,

Design Chart

From the principle of operation of the peak-holding optimalizing control it is seen that the most important quantity to be specified for its design is the critical indicated difference c between the reading of the optical output sensing instrument and the output itself. By definition, c is the difference of the maximum of the actual output y and the value of y at the tripping instant of the input drive. The instant of reversing the input drive is typified by $t/T = \frac{1}{2}$. If the corresponding instant of maximum y is t^*, then the critical indicated difference c is calculated as

$$c = y\left(\frac{t^*}{T}\right) - y\left(\frac{1}{2}\right) \qquad (31)$$

by using either one of the Eqs. (27), (28) and (29). Since the instant of input drive reversal must come after the instant of maximum output, $t^*/T < \frac{1}{2}$.

To determine t^*, we may use the condition of zero slope, i.e., $dy/dt = 0$. Then Eq. (27a) gives

$$-\left[\frac{t^*}{T} - \left(\frac{\tau_o}{T} + \frac{\tau_i}{T}\right)\right] + \frac{(\tau_o/T)}{2\left(\frac{\tau_o}{T} - \frac{\tau_i}{T}\right)}\left[\frac{2\left(\frac{\tau_o}{T}\right)^2 \tanh\frac{T}{2\tau_o}}{\left(\frac{\tau_o}{T} - \frac{\tau_i}{T}\right)\left(2\frac{\tau_o}{T} - \frac{\tau_i}{T}\right)} - 1\right]\frac{e^{-\frac{t^*/T}{\tau_o/T}}}{\sinh\frac{T}{2\tau_o}}$$

$$+\left[1 - \left(\frac{t^*}{T}\right)\left(\frac{T}{\tau_o}\right) - \frac{\tau_i/T}{\frac{\tau_o}{T} - \frac{\tau_i}{T}}\right]\frac{\left(\frac{\tau_i}{T}\right)^2 e^{-\frac{t^*/T}{\tau_i/T}}}{\left(\frac{\tau_o}{T} - \frac{\tau_i}{T}\right)\cosh\frac{T}{2\tau_o}} - \frac{\left(\frac{\tau_i}{T}\right)^2 e^{-\frac{2t^*/T}{\tau_i/T}}}{\left(2\frac{\tau_o}{T} - \frac{\tau_i}{T}\right)\cosh^2\frac{T}{2\tau_o}} = 0 \qquad (32)$$

This transcendental equation for t^*/T may be solved by iteration. For instance for small τ_o/T and τ_i/T, $t^*/T \cong (\tau_o + \tau_i)/T$. The results of calculation are shown in Fig. 9, which shows that t^*/T is almost only a function of $(\tau_o + \tau_i)/T$ with minor modifications from the parameter τ_o/τ_i, the ratio of characteristic times of the output linear group and the input linear group.

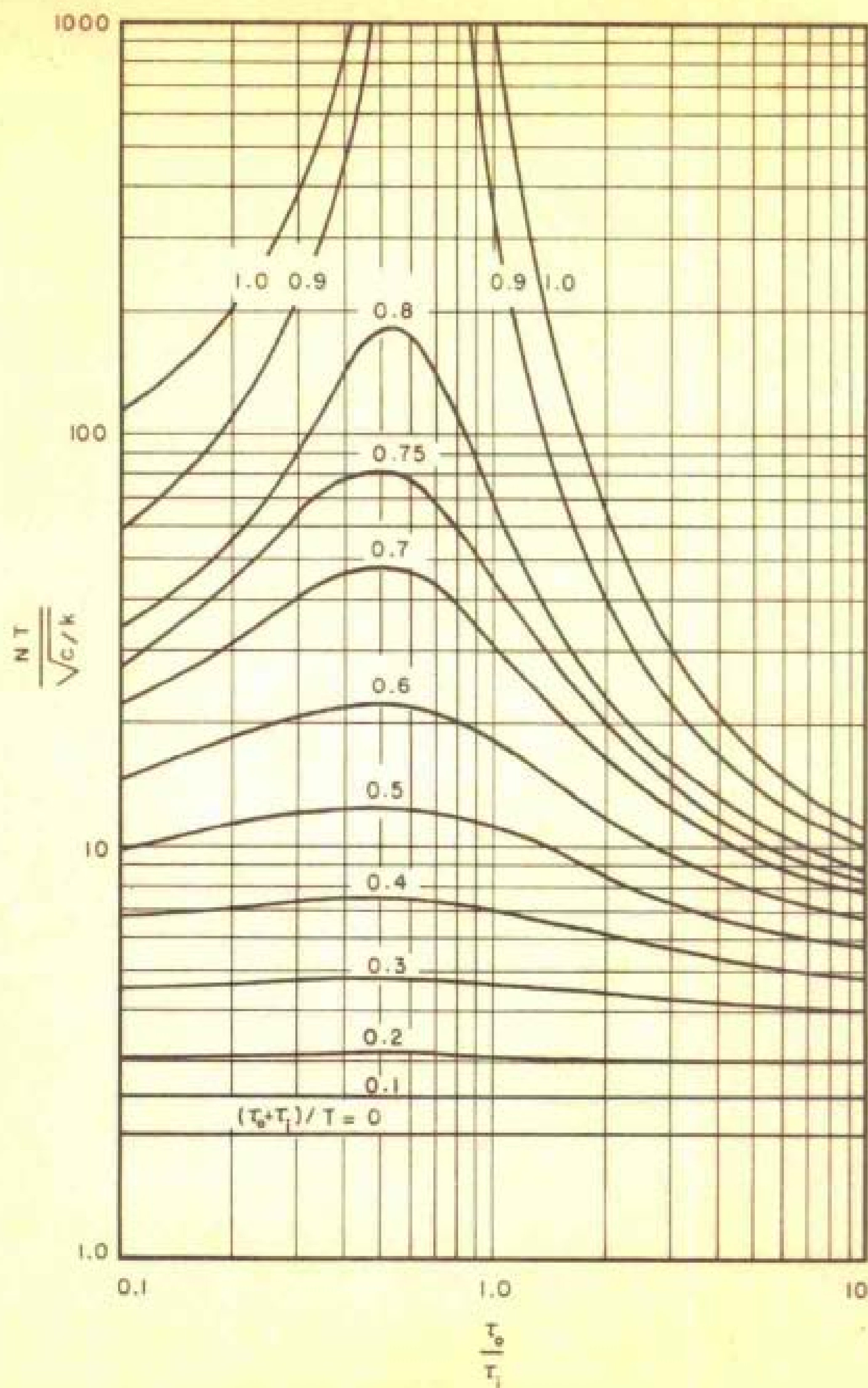
16

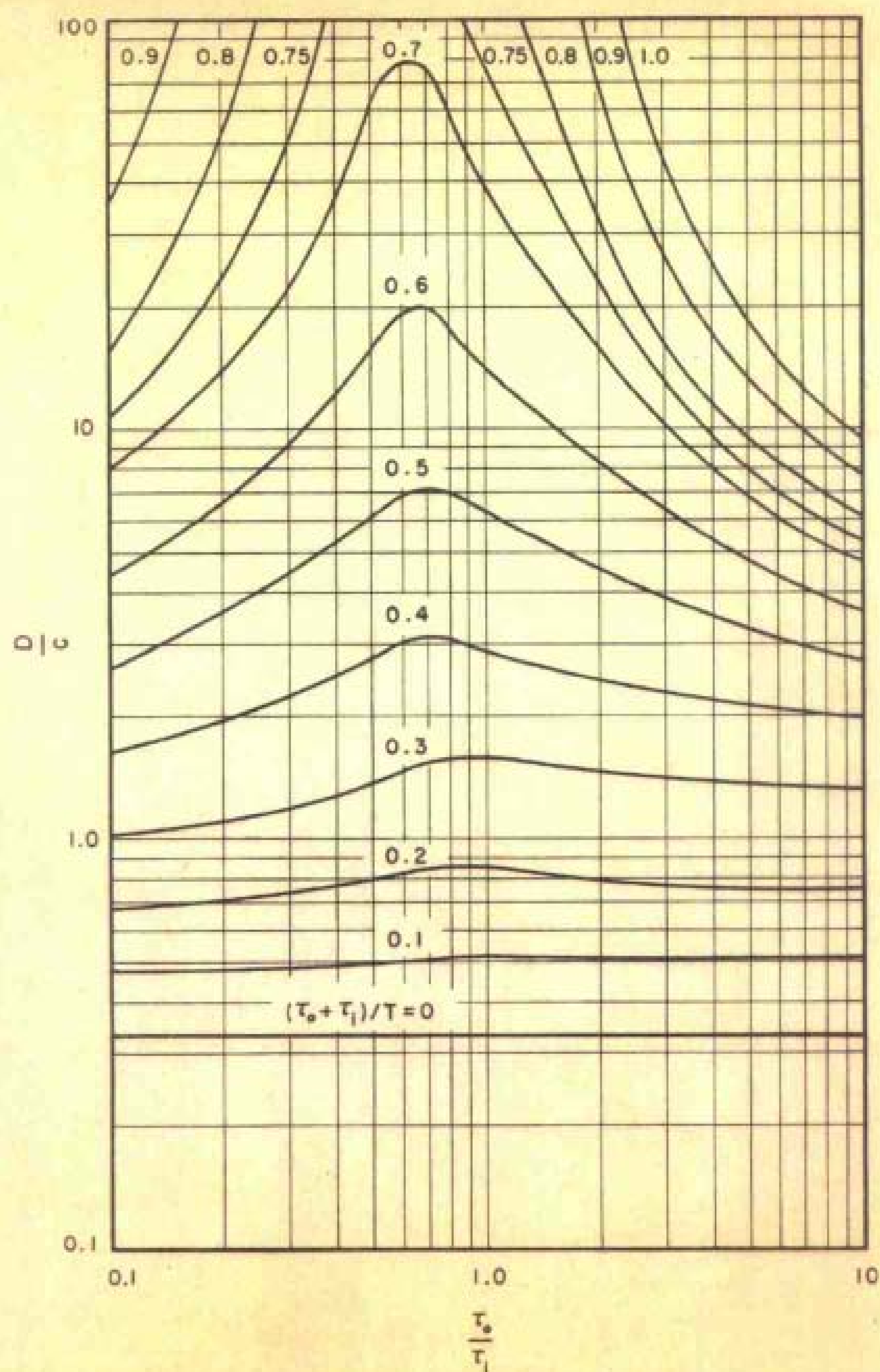
With t^*/T determined, Eq. (21) gives c by substituting Eq. (27a). However the specified quantities of an optimalizing control are h, the characteristics of the controlled system, and τ_i, τ_o, the characteristics of the linear groups. From considerations on the noise interference, the designer can make an appropriate choice of the period T and the critical indicated difference c for input drive reversal. Therefore the quantities which the designer wish to know, after he has the values of h, τ_i, τ_o, T and c, are N, the input drive speed, and D, the hunting loss. Thus the result of calculation with Eq. (31) should be written as follows:

$$\frac{TN^*}{\sqrt{ch}} = \frac{1}{\sqrt{[\text{illegible}]}} \tag{33}$$

When N is determined, Eq. (30) then gives the hunting loss D.

Figs. 10 and 11 are the design charts for peak-holding optimalizing control computed from the equations of the preceding analysis. Fig. 10 gives $TN/\sqrt{ch}$ as a function of $(\tau_o+\tau_i)/T$ with τ_o/τ_i as parameter. Fig. 11 gives D/c again as a function of $(\tau_o+\tau_i)/T$ with τ_o/τ_i as parameter.


1000
100
10
1.0
0.1
1.0
10
1.0 0.9
0.9 1.0
0.8
0.75
0.7
0.6
0.5
0.4
0.3
0.2
0.1
$(\tau_0+\tau_1)/T = 0$
$\frac{NT}{\sqrt{c/k}}$
$\frac{\tau_0}{\tau_1}$


100
10
1.0
0.1
0.9
0.8
0.75
0.7
0.75
0.8
0.9
1.0
0.6
0.5
0.4
0.3
0.2
0.1
(τ₀ + τ₁)/T = 0
D/c
0.1
1.0
10
τ₀/τ₁

3.6

Noise Filtering in a Guidance System

制导系统的噪声滤波

这是作者对火箭的制导系统噪声滤波所做的一个完整分析的部分手稿，共有7页。这一工作并未形成正式论文，工作时间不详。

在任何一个工程系统中都存在噪声和干扰，甚至于“完善”的系统也有热扰动。只有在信号比干扰强得多的情况下，噪声和干扰对控制系统的影响才能忽略不计。在自动寻求最优运转点系统的设计中，为了减少噪声和干扰使输出信号变模糊的程度，减少输出的搜索损失，需要对噪声进行处理。一个有效的方法是在控制系统中引入一个过滤噪声的装置。

作者的这一工作是将噪声滤波的方法应用到火箭的制导系统中，这里选印手稿的开头和结尾两部分，共7页。前面2页是开头部分，把噪声过滤提成一个变分问题，即寻求过滤器应具备什么样的特性(由过滤器对单位脉冲的响应函数h表示)，才能使误差达到最小，从而在最大程度上滤掉噪声而显示有用信号。后面5页是结尾部分，讨论了一般的线性过滤处理白噪声的情况，说明经过过滤能得到肯定的效果。这些结果被作者收入《工程控制论》一书。

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Noise Filtering in a Guidance System

1) Let the error ε be given by the following integral

$$\varepsilon = \int_0^T [\alpha(t)x(t) + \beta(t)y(t)]\,dt \qquad (1)$$

where

t = time

T = flight duration of normal trajectory

$\alpha(t)$ = given function

$\beta(t)$ = " "

$x(t)$ = control

$y(t)$ = true tracking information, unknown arbitrary function

If $n(t)$ is the noise with

$$\langle n(t)\rangle = 0 \qquad (2)$$

$$\langle n(t)n(t+\tau)\rangle = \langle n^2\rangle R(\tau) \qquad (3)$$

then the information received is

$$y(t) + n(t).$$

Let

$$\alpha(t)x(t) + \beta(t)\left[y(t) + n(t) + \int_0^t \{y(\tau)+n(\tau)\}h(t,\tau)\,d\tau\right] = 0 \qquad (4)$$

Then

$$\varepsilon = -\int_0^T \beta(t)\left[n(t) + \int_0^t \{y(\tau)+n(\tau)\}h(t,\tau)\,d\tau\right]dt \qquad (5)$$

Due to (2),

$$\langle\varepsilon\rangle = -\int_0^T \beta(t)\left[\int_0^t y(\tau)h(t,\tau)\,d\tau\right]dt \qquad (6)$$

Or

$$\langle\varepsilon\rangle = -\int_0^T y(\tau)\left[\int_\tau^T \beta(t)h(t,\tau)\,dt\right]d\tau \qquad (7)$$

Hence the condition

$$\langle\varepsilon\rangle = 0 \qquad (8)$$

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is

$$\boxed{\int_t^T \beta(\tau) h(\tau, t)\, d\tau = 0} \tag{9}$$

Then, Eq. (5) gives

$$\varepsilon = -\int_0^T \beta(t)\left[n(t) + \int_0^t n(\tau) h(t,\tau)\, d\tau\right] dt \tag{10}$$

Or

$$\varepsilon^2 = \int_0^T \beta(t)\left[n(t) + \int_0^t n(\tau) h(t,\tau)\, d\tau\right] dt \cdot \int_0^T \beta(t')\left[n(t') + \int_0^{t'} n(\tau') h(t',\tau')\, d\tau'\right] dt'$$

$$= \int_0^T dt \int_0^T dt'\, \beta(t)\beta(t')\, n(t)\, n(t') + 2\int_0^T dt \int_0^T dt'\, \beta(t)\beta(t') \int_0^{t'} n(t) n(\tau') h(t',\tau')\, d\tau'$$

$$+ \int_0^T dt \int_0^T dt'\, \beta(t)\beta(t') \int_0^t d\tau \int_0^{t'} d\tau'\, n(\tau)\, n(\tau')\, h(t,\tau)\, h(t',\tau') \tag{11}$$

Thus

$$\boxed{\begin{aligned} \langle \varepsilon^2 \rangle / \langle n^2 \rangle = &\int_0^T dt \int_0^T dt'\, \beta(t)\beta(t')\, R(t'-t) \\ &+ 2\int_0^T dt \int_0^T dt'\, \beta(t)\beta(t') \int_0^{t'} R(\tau'-t)\, h(t',\tau')\, d\tau' \\ &+ \int_0^T dt \int_0^T dt'\, \beta(t)\beta(t') \int_0^t d\tau \int_0^{t'} d\tau'\, R(\tau'-\tau)\, h(t,\tau)\, h(t',\tau') \end{aligned}} \tag{12}$$

according to Eq. (3).

The problem: Given $\beta(t)$, $R(\tau)$, determine $h(t,\tau)$ such that Eq. (9) is satisfied and $\langle \varepsilon^2 \rangle / \langle n^2 \rangle$ of Eq. (12) is a minimum.

II) That our problem makes sense can be seen as follows:

Ex. 1. If $\beta(t)$ = constant, then Eq. (9) will be satisfied by a $h(\tau, t)$ antisymmetric in τ with respect to $\tau = T - \left(\frac{T-t}{2}\right)$

$$H(s) = -\frac{1}{2\pi i\,\psi(s)}\int_0^{\infty} e^{-st}\,dt \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Phi_n\left(\frac{s'}{i}\right)B(-s')B(s') + \frac{P_n(-s')}{(-s')^{n+1}}B(s')}{\psi(-s')}\, e^{s't}\,ds' \tag{37}$$

Let $\psi_n(s)\psi_n(-s) = \Phi_n\left(\frac{s}{i}\right)$

$$\psi(s) = \psi_n(s)\,B(-s)$$

$$\psi(-s) = \psi_n(-s)\,B(s)$$

$$H(s) = -\frac{1}{2\pi i\,\psi_n(s)B(-s)}\int_0^{\infty} e^{-st}\,dt \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Phi_n\left(\frac{\eta}{i}\right)B(-\eta) + \frac{P_n(-\eta)}{(-\eta)^{n+1}}}{\psi_n(-\eta)}\, e^{\eta t}\,d\eta \tag{38}$$

The constants $a_0, \dots a_n$ are finally determined by Eq. (20). This then complete the problem of optimum filtering with linear circuit of constant coefficients.

VII) Now return to the problem of general linear filtering in case of white noise. Then

$$\int_0^T \tau^m d\tau \int_\tau^T \beta(t) h(t,\tau)\,dt = 0, \quad m = 0, 1, 2, \cdots n \tag{39}$$

and minimize

$$\langle \varepsilon^2\rangle / \langle n^2\rangle = \int_0^T \beta^2(t)\,dt + 2\int_0^T \beta(t)\,dt \int_t^T \beta(t')h(t',t)\,dt'$$

$$+ 2\int_0^T \beta(t)\,dt \int_0^t \beta(t')\,dt' \int_0^{t'} h(t,\tau)h(t',\tau)\,d\tau$$

This is equivalent to minimize

$$I = \int_0^T \beta^2(t)\,dt + 2\int_0^T \beta(\tau)\,d\tau \int_\tau^T \beta(t)\,h(t,\tau)\,dt$$

$$+ 2\int_0^T \beta(t)\,dt \int_0^t \beta(t')\,dt' \int_0^{t'} h(t,\tau)\,h(t',\tau)\,d\tau$$

$$+ 2\int_0^T (a_0 + a_1\tau + a_2\tau^2 + \cdots + a_n\tau^n)\,d\tau \int_\tau^T \beta(t)\,h(t,\tau)\,dt$$

$$\delta I = 2\int_0^T \beta(\tau)\,d\tau \int_\tau^T \beta(t)\,\delta h(t,\tau)\,dt$$

$$+ 2\int_0^T (a_0 + a_1\tau + a_2\tau^2 + \cdots + a_n\tau^n)\,d\tau \int_\tau^T \beta(t)\,\delta h(t,\tau)\,dt$$

$$+ 2\int_0^T \beta(t)\,dt \int_0^t \beta(t')\,dt' \int_0^{t'} \delta h(t,\tau)\,h(t',\tau)\,d\tau$$

$$+ 2\int_0^T \beta(t)\,dt \int_0^t \beta(t')\,dt' \int_0^{t'} h(t,\tau)\,\delta h(t',\tau)\,d\tau$$

But $\displaystyle\int_0^T \beta(t)\,dt \int_0^t \beta(t')\,dt' \int_0^{t'} \delta h(t,\tau)\,h(t',\tau)\,d\tau$

$$= \int_0^T d\tau \int_\tau^T \beta(t')\,h(t',\tau)\,dt' \int_{t'}^T \beta(t)\,\delta h(t,\tau)\,dt$$

$$= \int_0^T d\tau \int_\tau^T \beta(t')\,h(t',\tau)\,dt' \int_\tau^T \beta(t)\,\delta h(t,\tau)\,dt - \int_0^T d\tau \int_\tau^T \beta(t')\,h(t',\tau)\,dt' \int_\tau^{t'} \beta(t)\,\delta h(t,\tau)\,dt$$

But $\displaystyle\int_0^T \beta(t)\,dt \int_0^t \beta(t')\,dt' \int_0^{t'} h(t,\tau)\,\delta h(t',\tau)\,d\tau$

$$= \int_0^T \beta(t')\,dt' \int_0^{t'} \beta(t)\,dt \int_0^t h(t',\tau)\,\delta h(t,\tau)\,d\tau$$

$$= \int_0^T d\tau \int_\tau^T \beta(t')\,h(t',\tau)\,dt' \int_\tau^{t'} \beta(t)\,\delta h(t,\tau)\,dt$$

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Hence

$$\delta I = 2\int_0^T d\tau\left[\beta(\tau)+a_0+a_1\tau+\cdots+a_n\tau^n+\int_\tau^T\beta(t)h(t,\tau)dt\right]\int_\tau^T\beta(t)\,\delta h(t,\tau)dt$$

Thus $\delta I = 0$ for any $\delta h(t,\tau)$ requires

$$\beta(\tau)+a_0+a_1\tau+\cdots\cdots+a_n\tau^n+\int_\tau^T\beta(t)h(t,\tau)dt=0 \qquad (40)$$

Then Eq. (39) can be written as

$$\int_0^T \tau^m(a_0+a_1\tau+\cdots+a_n\tau^n)d\tau = -\int_0^T\tau^m\beta(\tau)d\tau$$

Or

$$\frac{T^{m+1}}{(m+1)}\left\{a_0+\frac{m+1}{m+2}Ta_1+\frac{m+1}{m+3}T^2a_2+\cdots+\frac{m+1}{m+n+1}T^na_n\right\}=-\int_0^T\tau^m\beta(\tau)d\tau$$

Or

$$a_0+\left(\frac{m+1}{m+2}T\right)a_1+\left(\frac{m+1}{m+3}T^2\right)a_2+\cdots+\left(\frac{m+1}{m+n+1}T^n\right)a_n=-\frac{m+1}{T^{m+1}}\int_0^T\tau^m\beta(\tau)d\tau$$

$$m=0,1,2,\cdots\cdots n. \qquad (41)$$

$n+1$ eq. for a's.

If $n=1$,

$$a_0+\frac{1}{2}Ta_1=-\frac{1}{T}\int_0^T\beta(t)dt$$

$$a_0+\frac{2}{3}Ta_1=-\frac{2}{T^2}\int_0^T t\beta(t)dt$$

$$a_0=-\frac{2}{T}\int_0^T\left(2-3\frac{t}{T}\right)\beta(t)dt$$

$$\frac{1}{6}Ta_1=\frac{1}{T}\left\{\int_0^T\left(1-2\frac{t}{T}\right)\beta(t)dt\right\}$$

$$Ta_1=\frac{6}{T}\left\{\int_0^T\left(1-2\frac{t}{T}\right)\beta(t)dt\right\}$$

$$-\frac{1}{2}Ta_1=\frac{3}{T}\left\{\int_0^T\left(-1+2\frac{t}{T}\right)\beta(t)dt\right\}$$

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$$\beta(\tau) - \left(\frac{2}{T}\int_0^T \left(2 - 3\frac{t}{T}\right)\beta(t)\,dt\right) + \frac{\tau}{T}\left(\frac{6}{T}\int_0^T \left(1 - 2\frac{t}{T}\right)\beta(t)\,dt\right) + \int_\tau^T \beta(t)\,h(t,\tau)\,dt = 0.$$

In any event, it is evident that the coefficients a_i are determined once $\beta(t)$ is specified. Let us call the polynomial

$$\boxed{\pi(\tau) = a_0 + a_1\tau + \cdots + a_n\tau^n}$$

Then

$$\int_\tau^T \beta(t)\,h(t,\tau)\,dt = -\beta(\tau) - \pi(\tau)$$

and

$$\langle\varepsilon^2\rangle/\langle n^2\rangle = \int_0^T \beta^2(t)\,dt + 2\int_0^T \beta(t)\{-\beta(t) - \pi(t)\}\,dt$$

$$+ \int_0^T \beta(t)\,dt \int_0^t \beta(t')\,dt' \int_0^{t'} h(t,\tau)\,h(t',\tau)\,d\tau$$

$$+ \int_0^T \beta(t')\,dt' \int_0^{t'} \beta(t)\,dt \int_0^{t} h(t,\tau)\,h(t',\tau)\,d\tau$$

$$= \int_0^T \beta^2(t)\,dt + 2\int_0^T \beta(t)\{-\beta(t) - \pi(t)\}\,dt$$

$$+ \int_0^T d\tau \int_\tau^T \beta(t')\,h(t',\tau)\,dt' \left\{\int_\tau^T \beta(t)\,h(t,\tau)\,d\tau - \int_\tau^{t'} \beta(t)\,h(t,\tau)\,d\tau\right\}$$

$$+ \int_0^T d\tau \int_\tau^T \beta(t')\,h(t',\tau)\,dt' \int_\tau^{t'} \beta(t)\,h(t,\tau)\,d\tau$$

$$= \int_0^T \left[\beta^2(t) - 2\beta(t)\{\beta(t) + \pi(t)\} + \{\beta(t) + \pi(t)\}^2\right] dt$$

$$= \boxed{\int_0^T \pi^2(t)\,dt = \langle\varepsilon^2\rangle/\langle n^2\rangle}$$

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Eq. (39) and (40) also give

$$\int_0^T \pi^2(t)\,dt = -\int_0^T \beta(t)\,\pi(t)\,dt.$$

Or

$$\int_0^T \pi(t)\{\beta(t)+\pi(t)\}\,dt = 0. \qquad (42)$$

Then

$$\int_0^T \beta^2(t)\,dt - \int_0^T \pi^2(t)\,dt = \text{improvement of } \langle \varepsilon^2\rangle/\langle x^2\rangle \text{ by}$$

fitting,

$$= \int_0^T \{\beta(t)+\pi(t)\}\{\beta(t)-\pi(t)\}\,dt$$

But due to Eq. (42),

$$\text{Improvement} = \int_0^T \{\beta(t)+\pi(t)\}^2\,dt \geq 0. \quad \text{proof!}$$

物 理 力 学

4.1
The Properties of Pure Liquids
液体特性

液体特性的研究是作者于1952年完成的。现存文稿包括：(1) 论文手稿(30页)，(2) S. S. Penner (潘纳)提供的文献单，(3) 收集的实验资料、数据及其处理，(4) 量子效应的讨论，(5) 题为 "Lennard - Jones - Devonshire Theory" (林纳德 - 琼斯 - 德文沙理论) 实为论文的理论推导(13页)，(6) 题为 "Buckingham Potential" (伯金汉势) 实为用伯金汉势作为分子势的理论推导等。这里只选印论文手稿之首页及林纳德 - 琼斯 - 德文沙理论的推导13页。

论文 " The Properties of Pure Liquids" (液体特性) 正式发表于《J. Amer. Rocket Society》(美国火箭学会学报)，Vol. 23，No. 1，17 - 24 (1953)，后来作者将其主要内容收入他所编著的《物理力学讲义》(1962)之中，构成其第九章"液体与稠密气体"的内容之一部分。在论文的序言部分着意指出，其追求不在于完全从原子分子层次出发给出计算液体性质的严格方法，而是把理论当做一种构架，就像传统的固体力学和流体力学中的"量纲分析"一样，将液体性质参数关联成相互依存的无量纲量，用部分实验结果将其定量化。例如把液体的压缩系数与沸点、密度和分子量关联起来。一种液体的沸点、密度和分子量易于查找或测定，而压缩系数并非总可

查到的。用本文中的方法则可使工程师们在做出定量估计时有可靠的依据。

这里给出的是该论文理论框架的推导过程。原稿的标题是“Lennard - Jones - Devonshire Theory”（林纳德 - 琼斯 - 德文沙理论）。计算初始的物理模型是当时最为成功的“笼子”理论。作者的工作是从笼子模型配分函数统计表达式的 g 积分进行渐近展开近似开始的，从而可以得到关于 g 的一个解析表达式（当时物理化学家们的主要努力是集中到对 g 求一个数值表的道路）。手稿中有作者用 Buckingham Potential（伯金汉势）在同样模型下进行的推导，但是他在论文发表时并未使用。手稿中包括作者当时收集的大量实验资料，显然，这部分内容对论文的成功是必不可少的。

Draft, one copy! 1

The Properties of Pure Liquids*

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Summary

By a semi-empirical approach, a method is found to calculate the specific heat of a normal pure liquid at constant pressure from the specific heat of the gaseous state at the same temperature. It is also found that the coefficient of thermal expansion, the compressibility and the velocity of sound of the liquid can be calculated accurately if the density, the molecular weight and the normal boiling temperature of the liquid at atmospheric pressure are known. Finally a method of computing the thermal conductivity of all liquids, except liquid metals, from compressibility and density is developed. For normal liquids, the thermal conductivity can again be determined if only the normal boiling temperature, density and molecular weight are known.

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Lennard-Jones, Devonshire Theory

F.-G. p. 336 —

The equation of states is given by

$$P = \frac{\partial}{\partial V}\left\{ kT \log(2\pi\gamma g V) + \Lambda^* \left[1.2 \left(\frac{V^*}{V}\right)^2 - 0.5 \left(\frac{V^*}{V}\right)^4 \right] \right\}$$

where $\gamma = a^3/V$, a is the average distance between nearest neighbors, V is the volume occupied per molecule. For a face centered cubic lattice, the edge of an elementary cubic cell containing four molecules is equal to $a\sqrt{2}$, $V = \frac{1}{4}(a\sqrt{2})^3 = \frac{1}{\sqrt{2}}a^3$, or $\gamma = \sqrt{2}$.

$$g = \int_0^{\frac{1}{4}} y^{\frac{1}{2}} \exp\left\{ -\frac{\Lambda^*}{kT}\left(\frac{V^*}{V}\right)^4 l(y) + 2\frac{\Lambda^*}{kT}\left(\frac{V^*}{V}\right)^2 m(y) \right\} dy$$

where

$$\Lambda^* = z E_m$$

and z is the coördination number, for face centered cubic lattice it is 12. E_m is minimum potential for bi-molecular interaction. In fact, the bi-molecular interaction is given by

$$\varepsilon(r) = E_m \left[\left(\frac{r^*}{r}\right)^{12} - 2\left(\frac{r^*}{r}\right)^6 \right]$$

$$= 4E_m \left[\left(\frac{r_0}{r}\right)^{12} - \left(\frac{r_0}{r}\right)^6 \right] \qquad 2r_0^6 = r^{*6}$$

$$V^* = \frac{V}{a^3} r^{*3} = \frac{1}{\gamma} r^{*3} = \frac{\sqrt{2}}{\gamma} r_0^3$$

and

$$l(y) = \frac{1 + 12y + 25.2y^2 + 12y^3 + y^4}{(1-y)^{10}} - 1, \qquad l(0) = 0$$

$$m(y) = \frac{1+y}{(1-y)^4} - 1, \qquad m(0) = 0$$

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First Approximation, $V \ll V^*$

$$(1-y)^{-10} = 1 + 10y + \frac{10\cdot 11}{2}y^2 + \cdots$$

$$l(y) = (1 + 12y + 25.2y^2 + \cdots)(1 + 10y + \frac{10\cdot 11}{2}y^2 + \cdots) - 1$$

$$= 22y + 201.2y^2 + \cdots$$

$$(1-y)^{-4} = 1 + 4y + 10y^2 + \cdots$$

$$m(y) = (1+y)(1 + 4y + 10y^2 + \cdots) - 1 = 5y + 14y^2 + \cdots$$

$$g = \int_0^{\frac{1}{4}} y^{\frac{1}{2}} \exp\left\{ -\frac{\Lambda^*}{kT}\left(\frac{V^*}{V}\right)^4 (22y + 201.2y^2 \cdots) + 2\frac{\Lambda^*}{kT}\left(\frac{V^*}{V}\right)^2 (5y + 14y^2 \cdots) \right\} dy$$

Let us introduce

$$\frac{\Lambda^*}{kT}\left(\frac{V^*}{V}\right)^4 22y = \xi, \qquad y = \frac{1}{22}\frac{kT}{\Lambda^*}\left(\frac{V}{V^*}\right)^4 \xi$$

$$-\frac{\Lambda^*}{kT}\left(\frac{V^*}{V}\right)^4 201.2y^2 = -\frac{201.2}{22^2}\frac{1}{\frac{\Lambda^*}{kT}\left(\frac{V^*}{V}\right)^4}\xi^2$$

$$2\frac{\Lambda^*}{kT}\left(\frac{V^*}{V}\right)^2 5y = \frac{10}{22}\frac{1}{\left(\frac{V^*}{V}\right)^2}\xi$$

$$y^{\frac{1}{2}}dy = \frac{2}{3}d(y^{3/2}) = \xi^{\frac{1}{2}}\left[\frac{1}{22}\frac{kT}{\Lambda^*}\left(\frac{V}{V^*}\right)^4\right]^{3/2} d\xi$$

$$g = \left[\frac{kT}{22\Lambda^*}\left(\frac{V}{V^*}\right)^4\right]^{3/2}\int_0^{\frac{22\Lambda^*}{4kT}\left(\frac{V^*}{V}\right)^4} \xi^{\frac{1}{2}} e^{-\xi}\left\{1 + \frac{10}{22}\frac{1}{\left(\frac{V^*}{V}\right)^2}\xi \cdots\right\} d\xi$$

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Approximation for Small V

$$l(y) = 22y + 201.2\,y^2 + \cdots$$

$$(1-y)^{-4} = 1 + 4y + 10y^2 + 20y^3 + 35\,y^4 + \cdots$$
$$y + 4y^2 + 10y^3 + 20y^4$$

$$m(y) = 5y + 14\,y^2 + 30\,y^3 + 55\,y^4 + \cdots$$

$$g = \int_0^{\frac{1}{4}} y^{\frac{1}{2}} \exp\left\{ -\frac{\Lambda^*}{kT}\left(\frac{V^*}{V}\right)^4 (22y + 201.2\,y^2 + \cdots) + 2\frac{\Lambda^*}{kT}\left(\frac{V^*}{V}\right)^2 (5y + 14y^2 + 30y^3 + 55y^4 \cdots)\right\} dy$$

Now let us put

$$22\,\frac{\Lambda^*}{kT}\left(\frac{V^*}{V}\right)^4 y = \xi$$

$$g = \left\{\frac{1}{22}\,\frac{kT}{\Lambda^*}\left(\frac{V}{V^*}\right)^4\right\}^{3/2} \int_0^{\frac{22\Lambda^*}{4kT}\left(\frac{V^*}{V}\right)^4} \xi^{\frac{1}{2}} e^{-\xi} d\xi \cdot \exp\left\{\frac{5}{11}\left(\frac{V}{V^*}\right)^2 \xi - \frac{100.6}{242}\,\frac{kT}{\Lambda^*}\left(\frac{V}{V^*}\right)^4 \xi^2 + \frac{7}{121}\,\frac{kT}{\Lambda^*}\left(\frac{V}{V^*}\right)^2 \xi^2\right\}$$

$$-\frac{\Lambda^*}{kT}\left(\frac{V^*}{V}\right)^4 201.2\,y^2 = -\frac{201.2}{22^2}\,\frac{1}{\frac{\Lambda^*}{kT}\left(\frac{V^*}{V}\right)^4}\,\xi^2$$

$$2\,\frac{\Lambda^*}{kT}\left(\frac{V^*}{V}\right)^2 5y = \frac{10}{22}\,\frac{1}{\left(\frac{V^*}{V}\right)^2}\,\xi$$

$$2\,\frac{\Lambda^*}{kT}\left(\frac{V^*}{V}\right)^2 14y^2 = \frac{28}{22^2}\,\frac{1}{\frac{\Lambda^*}{kT}\left(\frac{V^*}{V}\right)^6}\,\xi^2$$

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But
$$\int_0^{\frac{22\Lambda^*}{4kT}\left(\frac{V^*}{V}\right)^4} \xi^{\frac{1}{2}} e^{-\xi} d\xi = \int_0^{\infty} \xi^{\frac{1}{2}} e^{-\xi} d\xi - \int_{\frac{22\Lambda^*}{4kT}\left(\frac{V^*}{V}\right)^4}^{\infty} \xi^{\frac{1}{2}} e^{-\xi} d\xi$$

$$\cong \Gamma\left(\tfrac{3}{2}\right)$$

$$g \cong \left[\frac{kT}{22\Lambda^*}\left(\frac{V}{V^*}\right)^4\right]^{3/2} \left[\Gamma\left(\tfrac{3}{2}\right) + \frac{10}{22}\left(\frac{V}{V^*}\right)^2 \Gamma\left(\tfrac{5}{2}\right) + \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2}\left(\frac{kT}{22\Lambda^*}\right)^{3/2} \left(\frac{V}{V^*}\right)^6 \left[1 + \frac{15}{22}\left(\frac{V}{V^*}\right)^2 + \cdots\right]$$

$$\log Vg = \log\left[\frac{\sqrt{\pi}}{2}\frac{\left(\frac{kT}{22\Lambda^*}\right)^{3/2}}{V^{*6}}\right] + 7\log V + \frac{15}{22}\left(\frac{V}{V^*}\right)^2 + \cdots$$

So
$$P = kT\left[\frac{7}{V} + \frac{15}{11}\frac{V}{V^{*2}}\right] - \Lambda^*\left[2.4\frac{V^{*2}}{V^3} - 1.0\frac{V^{*4}}{V^5}\right]$$

$$\boxed{P = \frac{kT}{V}\left[2\frac{\Lambda^*}{kT}\left(\frac{V^*}{V}\right)^4\left\{1 - 1.2\left(\frac{V}{V^*}\right)^2\right\} + 7 + \frac{15}{11}\frac{V^2}{V^{*2}}\right]}$$

$$\frac{\Lambda^*}{kT} = \frac{1}{2}\frac{E_n}{kT} = \left(\frac{\frac{1}{2}E_n}{k}\right)\Big/T$$

$$\sim 1+\frac{5}{11}\left(\frac{V}{V^*}\right)^2\xi-\frac{100.6}{242}\frac{kT}{\Lambda^*}\left(\frac{V}{V^*}\right)^4\xi^2+\frac{7}{121}\frac{kT}{\Lambda^*}\left(\frac{V}{V^*}\right)^6\xi^2\cdots$$

$$=1+\frac{5}{11}\left(\frac{V}{V^*}\right)^2\xi-\frac{100.6}{242}\frac{kT}{\Lambda^*}\left(\frac{V}{V^*}\right)^4\xi^2+\frac{7}{121}\frac{kT}{\Lambda^*}\left(\frac{V}{V^*}\right)^6\xi^2\cdots$$

$$+\frac{25}{242}\left(\frac{V}{V^*}\right)^4\xi^2-\frac{503}{2662}\frac{kT}{\Lambda^*}\left(\frac{V}{V^*}\right)^6\xi^3$$

$$+\frac{125}{6\cdot 1331}\left(\frac{V}{V^*}\right)^6\xi^3\cdots$$

$$f=\left(\frac{kT}{22\Lambda^*}\right)^{3/2}\left(\frac{V}{V^*}\right)^6\left[\Gamma\left(\frac{3}{2}\right)+\frac{5}{11}\left(\frac{V}{V^*}\right)^2\Gamma\left(\frac{5}{2}\right)+\frac{25}{242}\left(\frac{V}{V^*}\right)^4\left\{1-\frac{100.6}{25}\frac{kT}{\Lambda^*}\right\}\Gamma\left(\frac{7}{2}\right)\right.$$

$$\left.+\left(\frac{V}{V^*}\right)^6\left\{\frac{7}{121}\frac{kT}{\Lambda^*}\Gamma\left(\frac{7}{2}\right)-\frac{503}{2662}\frac{kT}{\Lambda^*}\Gamma\left(\frac{9}{2}\right)+\frac{125}{6\cdot 1331}\Gamma\left(\frac{9}{2}\right)\right\}\cdots\right]$$

$$=\frac{\sqrt{\pi}}{2}\left(\frac{kT}{22\Lambda^*}\right)^{3/2}\left(\frac{V}{V^*}\right)^6\left[1+\frac{15}{22}\left(\frac{V}{V^*}\right)^2+\frac{15}{4}\frac{25}{242}\left\{1-\frac{100.6}{25}\frac{kT}{\Lambda^*}\right\}\left(\frac{V}{V^*}\right)^4\right.$$

$$\left.+\frac{35\cdot 125}{16\cdot 1331}\left\{1-\frac{6\cdot 1331}{125}\frac{503}{2662}\frac{kT}{\Lambda^*}+\frac{6\cdot 1331}{125}\frac{2}{7}\frac{7}{121}\frac{kT}{\Lambda^*}\right\}\left(\frac{V}{V^*}\right)^6\cdots\right]$$

$$=\frac{\sqrt{\pi}}{2}\left(\frac{kT}{22\Lambda^*}\right)^{3/2}\left(\frac{V}{V^*}\right)^6\left[1+\frac{15}{22}\left(\frac{V}{V^*}\right)^2+\frac{15}{4}\frac{25}{242}\left\{1-4.024\frac{kT}{\Lambda^*}\right\}\left(\frac{V}{V^*}\right)^4\right.$$

$$\left.+\frac{35\cdot 125}{16\cdot 1331}\left\{1-\frac{1377}{125}\frac{kT}{\Lambda^*}\right\}\left(\frac{V}{V^*}\right)^6\cdots\right]$$

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$$\log gV = \log \frac{\sqrt{\pi}}{2}\left(\frac{kT}{22\Lambda^*}\right)^{3/2}\frac{1}{V^{*6}} + 7\log V$$

$$+ \frac{15}{22}\left(\frac{V}{V^*}\right)^2 + \frac{15}{4}\,\frac{25}{242}\left\{1 - 4.024\frac{kT}{\Lambda^*}\right\}\left(\frac{V}{V^*}\right)^4 + \frac{35\cdot 125}{16\cdot 1331}\left\{1 - \frac{1377}{125}\frac{kT}{\Lambda^*}\right\}\left(\frac{V}{V^*}\right)^6$$

$$- \frac{9\cdot 25}{8\cdot 121}\left(\frac{V}{V^*}\right)^4 - \frac{225\cdot 25}{88\cdot 242}\left\{1 - 4.024\frac{kT}{\Lambda^*}\right\}\left(\frac{V}{V^*}\right)^6$$

$$+ \frac{15^3}{3\cdot 22^3}\left(\frac{V}{V^*}\right)^6$$

$$= \log\frac{\sqrt{\pi}}{2}\left(\frac{kT}{22\Lambda^*}\right)^{3/2}\frac{1}{V^{*6}} + 7\log V + \frac{15}{22}\left(\frac{V}{V^*}\right)^2 + \frac{25}{4\cdot 242}\left\{6 - 15\cdot 4.024\frac{kT}{\Lambda^*}\right\}\left(\frac{V}{V^*}\right)^4$$

$$+ \frac{125}{16\cdot 1331}\left\{8 - \left(\frac{1377}{125}\times 35 - 4.024\times 45\right)\frac{kT}{\Lambda^*}\right\}\left(\frac{V}{V^*}\right)^6 \cdots$$

$$\frac{pV}{kT} = -2\frac{\Lambda^*}{kT}\left(\frac{V^*}{V}\right)^4\left[1 - 1.2\left(\frac{V}{V^*}\right)^2\right] + 7$$

$$+ \frac{15}{11}\left(\frac{V}{V^*}\right)^2 + \frac{25}{242}\left\{6 - 15\cdot 4.024\frac{kT}{\Lambda^*}\right\}\left(\frac{V}{V^*}\right)^4$$

$$+ \frac{6\cdot 125}{16\cdot 1331}\left\{8 - \left(\frac{1377}{125}\times 35 - 4.024\times 45\right)\frac{kT}{\Lambda^*}\right\}\left(\frac{V}{V^*}\right)^6 + \cdots$$

$$\int_0^x \xi^{\frac{1}{2}} e^{-\xi}\,d\xi, \qquad x \gg 1$$

$$= \int_0^\infty \xi^{\frac{1}{2}} e^{-\xi}\,d\xi - \int_x^\infty \xi^{\frac{1}{2}} e^{-\xi}\,d\xi = \Gamma\left(\tfrac{3}{2}\right) - \int_x^\infty \xi^{\frac{1}{2}} e^{-\xi}\,d\xi$$

$$= \Gamma\left(\tfrac{3}{2}\right) - \left[\left(-\xi^{\frac{1}{2}} e^{-\xi}\right)_x^\infty + \frac{1}{2}\int_x^\infty \xi^{-\frac{1}{2}} e^{-\xi}\,d\xi\right]$$

2

$$= \Gamma\left(\tfrac{3}{2}\right) - \left[x^{\frac{1}{2}} e^{-x} + \tfrac{1}{2} x^{-\frac{1}{2}} e^{-x} - \tfrac{1}{4} x^{-\frac{3}{2}} e^{-x} + \tfrac{3}{8} x^{-\frac{5}{2}} e^{-x} \cdots \right]$$

$$= \Gamma\left(\tfrac{3}{2}\right) - x^{\frac{1}{2}} e^{-x} \left[1 + \frac{1}{2x} - \frac{1}{4x^2} + \frac{3}{8x^3} \cdots \right] = \int_0^x \xi^{\frac{1}{2}} e^{-\xi} d\xi$$

$$\int_0^x \xi^{\frac{3}{2}} e^{-\xi} d\xi = \Gamma\left(\tfrac{5}{2}\right) - \int_x^\infty \xi^{3/2} e^{-\xi} d\xi$$

$$= \Gamma\left(\tfrac{5}{2}\right) - x^{3/2} e^{-x} \left[1 + \frac{3}{2x} + \frac{3}{4x^2} - \frac{3}{8x^3} \cdots \right]$$

$$\int_0^x \xi^{\frac{5}{2}} e^{-\xi} d\xi = \Gamma\left(\tfrac{7}{2}\right) - x^{5/2} e^{-x} \left[1 + \frac{5}{2x} + \frac{15}{4x^2} + \frac{15}{8x^3} \cdots \right]$$

$$\int_0^x \xi^{\frac{7}{2}} e^{-\xi} d\xi = \Gamma\left(\tfrac{9}{2}\right) - x^{7/2} e^{-x} \left[1 + \frac{7}{2x} + \frac{35}{4x^2} + \frac{105}{8x^3} + \cdots \right]$$

Therefore a more accurate value for g is

$$g \left(\frac{22\Lambda^8}{kT}\right)^{3/2} \left(\frac{V^*}{V}\right)^6 = \Gamma\left(\tfrac{3}{2}\right) - \left(\frac{22\Lambda^8}{4kT}\right)^{\frac{1}{2}} \left(\frac{V^*}{V}\right)^2 e^{-\frac{22\Lambda^8}{4kT}\left(\frac{V^*}{V}\right)^4} \left[1 + \frac{2kT}{22\Lambda^8}\left(\frac{V}{V^*}\right)^4 \cdots \right]$$

$$+ \frac{5}{11}\left(\frac{V}{V^*}\right)^2 \left\{ \Gamma\left(\tfrac{5}{2}\right) - \left(\frac{22\Lambda^8}{4kT}\right)^{\frac{3}{2}} \left(\frac{V^*}{V}\right)^6 e^{-\frac{22\Lambda^8}{4kT}\left(\frac{V^*}{V}\right)^4} \left[1 + \frac{6kT}{22\Lambda^8}\left(\frac{V}{V^*}\right)^4 \cdots \right] \right\}$$

$$+ \frac{25}{242}\left(\frac{V}{V^*}\right)^4 \left\{ \Gamma\left(\tfrac{7}{2}\right) - \left(\frac{22\Lambda^8}{4kT}\right)^{5/2} \left(\frac{V^*}{V}\right)^{10} e^{-\frac{22\Lambda^8}{4kT}\left(\frac{V^*}{V}\right)^4} \left[1 + \frac{10kT}{22\Lambda^8}\left(\frac{V}{V^*}\right)^4 \cdots \right] \right\}$$

The method is not right!

Another Method for Small V

$$(1-y)^{-10} = 1 + 10y + 55y^2 + 220y^3 + \cdots$$

$$\begin{aligned} l(y) = \ & 10y + 55y^2 + 220y^3 + \cdots \\ & 12y + 120y^2 + 660y^3 + \cdots \\ & 25.2y^2 + 252y^3 - \cdots \\ & 12y^3 - \cdots \\ \hline & 24y + 200.2y^2 + 1144y^3 + \cdots \end{aligned}$$

$$(1-y)^{-4} = 1 + 4y + 10y^2 + 20y^3 + \cdots$$

$$m(y) = 5y + 14y^2 + 30y^3 + \cdots$$

Let $\quad t = \frac{\Lambda^*}{kT}\left(\frac{V^*}{V}\right)^4$, then

$$-\frac{\Lambda^*}{kT}\left(\frac{V^*}{V}\right)^4 l(y) + 2\frac{\Lambda^*}{kT}\left(\frac{V^*}{V}\right)^2 m(y)$$

$$= -t\left[l(y) - 2\left(\frac{V}{V^*}\right)^2 m(y)\right]$$

$$= -t\left[\left\{24 - 10\left(\frac{V}{V^*}\right)^2\right\}y + \left\{200.2 - 28\left(\frac{V}{V^*}\right)^2\right\}y^2 + \left\{1144 - 60\left(\frac{V}{V^*}\right)^2\right\}y^3 + \cdots\right]$$

$$= -t \cdot \eta$$

Let $\quad y = a_1\eta + a_2\eta^2 + a_3\eta^3 + \cdots$

$$\eta = \left\{24 - 10\left(\frac{V}{V^*}\right)^2\right\}\left\{a_1\eta + a_2\eta^2 + a_3\eta^3 + \cdots\right\}$$

$$+ \left\{200.2 - 28\left(\frac{V}{V^*}\right)^2\right\}\left\{a_1^2\eta^2 + 2a_1a_2\eta^3 + \cdots\right\}$$

$$+ \left\{1144 - 60\left(\frac{V}{V^*}\right)^2\right\}a_1^3\eta^3 + \cdots$$

l

$$1 = \left\{24 - 10\left(\frac{V_0}{V}\right)^2\right\} a_1$$

$$0 = \left\{24 - 10\left(\frac{V_0}{V}\right)^2\right\} a_2 + \left\{200.2 - 28\left(\frac{V_0}{V}\right)^2\right\} a_1^2$$

$$0 = \left\{24 - 10\left(\frac{V_0}{V}\right)^2\right\} a_3 + \left\{200.2 - 28\left(\frac{V_0}{V}\right)^2\right\} 2a_1a_2 + \left\{1144 - 60\left(\frac{V_0}{V}\right)^2\right\} a_1^3$$

therefore

$$a_1 = \frac{1}{24 - 10\left(\frac{V_0}{V}\right)^2}$$

$$a_2 = -\frac{200.2 - 28\left(\frac{V_0}{V}\right)^2}{\left[24 - 10\left(\frac{V_0}{V}\right)^2\right]^3}$$

$$a_3 = \frac{2\left[200.2 - 28\left(\frac{V_0}{V}\right)^2\right]^2 - \left[1144 - 60\left(\frac{V_0}{V}\right)^2\right]\left[24 - 10\left(\frac{V_0}{V}\right)^2\right]}{\left[24 - 10\left(\frac{V_0}{V}\right)^2\right]^5}$$

when $y = \frac{1}{4}$,

$$\eta\left(\tfrac{1}{4}\right) = \frac{1 + 3 + \frac{25.2}{16} + \frac{3}{16} + \frac{1}{256}}{\left(\frac{3}{4}\right)^{10}} - 1 \sim 90$$

$$n\left(\tfrac{1}{4}\right) = \frac{\frac{5}{4}}{\left(\frac{3}{4}\right)^4} - 1 = \frac{5}{4}\frac{4}{3}\frac{4}{3}\frac{4}{3}\frac{4}{3} - 1 = \frac{320}{81} - 1 = 2.95$$

$$g = \frac{2}{3}\int_0^x d\left(y^{3/2}\right) e^{-t\eta}$$

$$y^{3/2} = (a_1\eta)^{3/2}\left[1 + \frac{a_2}{a_1}\eta + \frac{a_3}{a_1}\eta^2 + \cdots\right]^{3/2}$$

$$= (a_1\eta)^{3/2}\left[1 + \frac{3}{2}\frac{a_2}{a_1}\eta + \frac{3}{2}\frac{a_3}{a_1}\eta^2 + \cdots + \frac{3}{8}\left(\frac{a_2}{a_1}\right)^2\eta^2 + \cdots\right]$$

2

$$\rho^{3/2} = a_1^{3/2}\left[\eta^{3/2} + \frac{3}{2}\frac{a_2}{a_1}\eta^{5/2} + \frac{3}{2}\left\{\frac{a_3}{a_1} + \frac{1}{4}\left(\frac{a_2}{a_1}\right)^2\right\}\eta^{7/2} + \cdots\right]$$

$$\frac{2}{3}d(\rho^{3/2}) = a_1^{3/2}\left[\eta^{1/2} + \frac{5}{2}\frac{a_2}{a_1}\eta^{3/2} + \frac{7}{2}\left\{\frac{a_3}{a_1} + \frac{1}{4}\left(\frac{a_2}{a_1}\right)^2\right\}\eta^{5/2} + \cdots\right]d\eta$$

$$\rho = a_1^{3/2}\left[t^{-3/2}\Gamma\left(\frac{3}{2}\right) + \frac{5}{2}\frac{a_2}{a_1}t^{-5/2}\Gamma\left(\frac{5}{2}\right) + \frac{7}{2}\left\{\frac{a_3}{a_1} + \frac{1}{4}\left(\frac{a_2}{a_1}\right)^2\right\}t^{-7/2}\Gamma\left(\frac{7}{2}\right) + \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2}\left(\frac{a_1}{t}\right)^{3/2}\left[1 + \frac{3\cdot 5}{2\cdot 2}\frac{a_2}{a_1}\frac{1}{t} + \frac{3\cdot 5\cdot 7}{2\cdot 2\cdot 2}\left\{\frac{a_3}{a_1} + \frac{1}{4}\left(\frac{a_2}{a_1}\right)^2\right\}\frac{1}{t^2} + \cdots\right]$$

$$\log\rho = \log\frac{\sqrt{\pi}}{2} + \frac{3}{2}\log\left(\frac{a_1}{t}\right) + \frac{3\cdot 5}{2\cdot 2}\frac{a_2}{a_1}\frac{1}{t} + \frac{3\cdot 5\cdot 7}{2\cdot 2\cdot 2}\left\{\frac{a_3}{a_1} + \frac{1}{4}\left(\frac{a_2}{a_1}\right)^2\right\}\frac{1}{t^2}$$

$$- \left(\frac{3\cdot 5}{2\cdot 2}\right)^2\frac{1}{2}\left(\frac{a_2}{a_1}\right)^2\frac{1}{t^2}$$

$$= \log\frac{\sqrt{\pi}}{2} + \frac{3}{2}\log\left(\frac{a_1}{t}\right) + \frac{3\cdot 5}{2\cdot 2}\frac{a_2}{a_1}\frac{1}{t} + \frac{3\cdot 5\cdot 7}{2\cdot 2\cdot 2}\left\{\frac{a_3}{a_1} - \frac{2}{7}\left(\frac{a_2}{a_1}\right)^2\right\}\frac{1}{t^2}\cdots$$

$$= \log\frac{\sqrt{\pi}}{2} + \frac{3}{2}\log\frac{kT\left(\frac{\nu}{\nu_0}\right)^4}{\Lambda^2\left[24 - 10\left(\frac{\nu}{\nu_0}\right)^2\right]}$$

$$- \frac{3\cdot 5}{2\cdot 2}\frac{200.2 - 28\left(\frac{\nu}{\nu_0}\right)^2}{\left\{24 - 10\left(\frac{\nu}{\nu_0}\right)^2\right\}^2}\frac{kT}{\Lambda^2}\left(\frac{\nu}{\nu_0}\right)^4$$

$$+ \frac{3\cdot 5\cdot 7}{2\cdot 2\cdot 2}\left[\frac{{\color{red}12}\,2\left\{200.2 - 28\left(\frac{\nu}{\nu_0}\right)^2\right\}^2 - {\color{red}7}\left\{1144 - 60\left(\frac{\nu}{\nu_0}\right)^2\right\}\left\{24 - 10\left(\frac{\nu}{\nu_0}\right)^2\right\}}{{\color{red}7}\left\{24 - 10\left(\frac{\nu}{\nu_0}\right)^2\right\}^4} - \frac{2}{7}\cancel{\frac{\left\{200.2 - 28\left(\frac{\nu}{\nu_0}\right)^2\right\}^2}{\left\{24 - 10\left(\frac{\nu}{\nu_0}\right)^2\right\}^4}}\right]\times$$

$$\times\left(\frac{kT}{\Lambda^2}\right)^2\left(\frac{\nu}{\nu_0}\right)^8 + \cdots$$

10

$$\log\psi = \left\{\log\frac{\sqrt{\pi}}{2} + \frac{3}{2}\log\frac{kT}{\Lambda^* V^{*4}}\right\} + 7\log V$$

$$-\frac{3\cdot5}{2\cdot2}\frac{kT}{\Lambda^*}\left(\frac{V^*}{V}\right)^6 \frac{200.2-28\left(\frac{V^*}{V}\right)^2}{\left\{24-10\left(\frac{V^*}{V}\right)^2\right\}^2}$$

$$+\frac{3\cdot5\cdot7}{2\cdot2\cdot2}\left(\frac{kT}{\Lambda^*}\right)^2\left(\frac{V^*}{V}\right)^8 \frac{12\left\{200.2-28\left(\frac{V^*}{V}\right)^2\right\}^2 - 7\left\{1144-40\left(\frac{V^*}{V}\right)^2\right\}\left\{24-10\left(\frac{V^*}{V}\right)^2\right\}}{7\left\{24-10\left(\frac{V^*}{V}\right)^2\right\}^4} + \cdots$$

$$= \left\{\qquad\right\} + 7\log V - \frac{3\cdot5}{2\cdot2}\frac{kT}{\Lambda^*}\left(\frac{V^*}{V}\right)^6 \frac{200.2-28\left(\frac{V^*}{V}\right)^2}{\left\{24-10\left(\frac{V^*}{V}\right)^2\right\}^2}$$

$$+\frac{3\cdot5\cdot\not{7}}{2\cdot2\cdot\not{7}}\left(\frac{kT}{\Lambda^*}\right)^2\left(\frac{V^*}{V}\right)^8 \frac{\overset{144384.24}{\cancel{288768.48}} - \overset{22187.2}{\cancel{44374.4}}\left(\frac{V^*}{V}\right)^2 + \overset{2604}{\cancel{5208}}\left(\frac{V^*}{V}\right)^4}{\not{7}\left\{24-10\left(\frac{V^*}{V}\right)^2\right\}^4}$$

$$\frac{PV}{kT} = 2\frac{\Lambda^*}{kT}\left(\frac{V^*}{V}\right)^6\left\{1-1.2\left(\frac{V^*}{V}\right)^2\right\} + 7$$

$$-\frac{3\cdot5}{2\cdot2}\frac{kT}{\Lambda^*}\left(\frac{V^*}{V}\right)^6\left[\frac{200.8-168\left(\frac{V^*}{V}\right)^2}{\left\{24-10\left(\frac{V^*}{V}\right)^2\right\}^2} + 40\left(\frac{V^*}{V}\right)^2\frac{200.2-28\left(\frac{V^*}{V}\right)^2}{\left\{24-10\left(\frac{V^*}{V}\right)^2\right\}^3}\right]$$

$$+\frac{3\cdot5}{2\cdot2}\left(\frac{kT}{\Lambda^*}\right)^2\left(\frac{V^*}{V}\right)^8\left[\frac{1155073.92 - 22187.2\left(\frac{V^*}{V}\right)^2 + 31248\left(\frac{V^*}{V}\right)^4}{\left\{24-10\left(\frac{V^*}{V}\right)^2\right\}^4}\right.$$

$$\left. + 80\left(\frac{V^*}{V}\right)^2\frac{144384.24 - 22187.2\left(\frac{V^*}{V}\right)^2 + 2604\left(\frac{V^*}{V}\right)^4}{\left\{24-10\left(\frac{V^*}{V}\right)^2\right\}^5}\right]$$

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From p. 9,

$$\log g = \log\sqrt{\frac{\pi}{2}} + \frac{3}{2}\log\frac{kT\left(\frac{V_0}{V}\right)^4}{\Lambda^*\left\{24-10\left(\frac{V_0}{V}\right)^2\right\}} - \frac{3\cdot 5}{2\cdot 2}\,\frac{300.2-21\left(\frac{V_0}{V}\right)^2}{\left\{24-10\left(\frac{V_0}{V}\right)^2\right\}^3}\,\frac{kT}{\Lambda^*}\left(\frac{V_0}{V}\right)^4 \cdots$$

$$F = -kT\log\frac{(2\pi mkT)^{3/2}}{h^3} - kT\log j(T) - kT$$

$$-\Lambda^*\left\{1.2\left(\frac{V_0}{V}\right)^2 - 0.5\left(\frac{V_0}{V}\right)^4\right\} - kT\left[\log 2\pi\gamma + \log V + \log g\right]$$

$$P = -\frac{\partial F}{\partial V} = -\frac{\Lambda^*}{V}\left\{2.4\left(\frac{V_0}{V}\right)^2 - 2\left(\frac{V_0}{V}\right)^4\right\}$$

$$+kT\left[\frac{1}{V} + \frac{3}{2}\,\frac{4}{V} - \frac{3}{2}\,\frac{(-)20\left(\frac{V_0}{V}\right)^2\frac{1}{V}}{24-10\left(\frac{V_0}{V}\right)^2}\right]\cdots\cdots$$

$24 \times 7 = 168$

$$P = 2\Lambda^*\frac{1}{V}\left(\frac{V_0}{V}\right)^4\left\{1-1.2\left(\frac{V_0}{V}\right)^2\right\} + \frac{kT}{V}\left[7 + 30\,\frac{\left(\frac{V_0}{V}\right)^2}{24-10\left(\frac{V_0}{V}\right)^2}\right]$$

$$= 2\Lambda^*\frac{1}{V}\left(\frac{V_0}{V}\right)^4\left\{1-1.2\left(\frac{V_0}{V}\right)^2\right\} + \frac{kT}{V}\,\frac{84-20\left(\frac{V_0}{V}\right)^2}{12-5\left(\frac{V_0}{V}\right)^2}$$

For T = constant,

$$V\frac{\partial P}{\partial V} = -2\Lambda^*\frac{1}{V}\left(\frac{V_0}{V}\right)^4\left\{+5-8.4\left(\frac{V_0}{V}\right)^2\right\} - \frac{kT}{V}\,\frac{84-20\left(\frac{V_0}{V}\right)^2}{12-5\left(\frac{V_0}{V}\right)^2}$$

$$+\frac{kT}{V}\,\frac{\left[-40\left(\frac{V_0}{V}\right)^2\right]\left[12-5\left(\frac{V_0}{V}\right)^2\right] - \left[84-20\left(\frac{V_0}{V}\right)^2\right]\left[-10\left(\frac{V_0}{V}\right)^2\right]}{\left[12-5\left(\frac{V_0}{V}\right)^2\right]^2}$$

$$-V\frac{\partial P}{\partial V} = 2\Lambda^*\frac{1}{V}\left(\frac{V_0}{V}\right)^4\left\{5-8.4\left(\frac{V_0}{V}\right)^2\right\} + \frac{kT}{V}\left\{\frac{84-20\left(\frac{V_0}{V}\right)^2}{12-5\left(\frac{V_0}{V}\right)^2} - 10\left(\frac{V_0}{V}\right)^2\frac{36}{\left[12-5\left(\frac{V_0}{V}\right)^2\right]^2}\right\}$$

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$$-\frac{V}{P}\frac{\partial P}{\partial V} = \frac{2\left(\frac{V^*}{V}\right)^4\left\{5-8.4\left(\frac{V^*}{V}\right)^2\right\} + \frac{kT}{\Lambda^*}\left\{\frac{84-20\left(\frac{V^*}{V}\right)^2}{12-5\left(\frac{V^*}{V}\right)^2} - \frac{360\left(\frac{V^*}{V}\right)^2}{\left[12-5\left(\frac{V^*}{V}\right)^2\right]^2}\right\}}{2\left(\frac{V^*}{V}\right)^4\left\{1-1.2\left(\frac{V^*}{V}\right)^2\right\} + \frac{kT}{\Lambda^*}\frac{84-20\left(\frac{V^*}{V}\right)^2}{12-5\left(\frac{V^*}{V}\right)^2}}$$

$$= \frac{5-8.4\left(\frac{V^*}{V}\right)^2}{1-1.2\left(\frac{V^*}{V}\right)^2}\left[1 + \frac{kT}{\Lambda^*}\frac{1}{2\left(\frac{V^*}{V}\right)^4}\left\{\frac{\frac{84-20\left(\frac{V^*}{V}\right)^2}{12-5\left(\frac{V^*}{V}\right)^2} - \frac{360\left(\frac{V^*}{V}\right)^2}{\left[12-5\left(\frac{V^*}{V}\right)^2\right]^2}}{5-8.4\left(\frac{V^*}{V}\right)^2} - \frac{\frac{84-20\left(\frac{V^*}{V}\right)^2}{12-5\left(\frac{V^*}{V}\right)^2}}{1-1.2\left(\frac{V^*}{V}\right)^2}\right\}\right]$$

$$E = -T^2\frac{\partial(F/T)}{\partial T}$$

$$= E^{int} + \frac{3}{2}kT - \Lambda^*\left\{1.2\left(\frac{V^*}{V}\right)^2 - 0.5\left(\frac{V^*}{V}\right)^4\right\}$$

$$+ kT^2\frac{\partial}{\partial T}(\log g)$$

$$= E^{int} + \frac{3}{2}kT - \Lambda^*\left\{1.2\left(\frac{V^*}{V}\right)^2 - 0.5\left(\frac{V^*}{V}\right)^4\right\}$$

$$+ kT^2\left[\frac{3}{2}\frac{1}{T} - \frac{3.5}{2.2}\frac{200.2-28\left(\frac{V^*}{V}\right)^2}{\left[24-10\left(\frac{V^*}{V}\right)^2\right]^2}\frac{k}{\Lambda^*}\left(\frac{V}{V^*}\right)^4 \cdots\right]$$

$$= E^{int} + 3kT - \Lambda^*\left\{1.2\left(\frac{V^*}{V}\right)^2 - 0.5\left(\frac{V^*}{V}\right)^4\right\}$$

$$- \frac{3.5}{2.2}\frac{50.05-7\left(\frac{V^*}{V}\right)^2}{\left[12-5\left(\frac{V^*}{V}\right)^2\right]^2}kT^2\frac{k}{\Lambda^*}\left(\frac{V}{V^*}\right)^4$$

$$\boxed{C_V = \frac{\partial E}{\partial T} = C_V^{int} + 3k - \frac{3.5}{2.2}k\frac{100.2-14\left(\frac{V^*}{V}\right)^2}{\left[12-5\left(\frac{V^*}{V}\right)^2\right]^2}\frac{kT}{\Lambda^*}\left(\frac{V}{V^*}\right)^2}$$

4.2

Thermodynamic Properties of Gas at High Temperature and Pressure

气体在高温高压下的热力学性质

高温高压气体的热力学性质研究完成于1954年。现存文稿包括：(1)题为"High Density Gas"（高密度气体）的论文前期的理论推导，(2)题为"Lennard - Jones and Devonshire Theory for Dense Gas"（稠密气体的林纳德——琼斯与德文沙理论）实为本研究最终所采用的半经验状态方程和热力学函数的理论推导，共13页，(3)收集的原始数据及其处理，(4)论文手稿共8页等。这里只选印论文手稿之首页及林纳德——琼斯与德文沙理的推导11页。

论文正式发表于《Jet Propulsion》(喷气推进学报)，Vol. 25, Part, Issue9, 471 - 478(1995)，其主要部分后来被作者编入《物理力学讲义》（科学出版社，1962)，构成该书之第九章"液体与稠密气体"的部分内容。论文中所讨论的高温高压气体状态正是工程技术上的凝聚炸药爆震所能达到的几十万大气压、几千度K的状态。这个问题既重要又困难。Ficket(菲克特)和Davis(戴维斯)在1979年的专著《Detonation》(有中译本：《爆轰》，原子能出版社，1988年版)中曾提到"液体和固体炸药CJ点附近区域中气体的性质，人们知之甚少"。书中评价了被采用的三种方法，即：Kistiakowsky - Wilson(基斯切可夫斯基 - 威尔逊)状态方程，Lennard - Jones - Devonshire理论，和Jacobs - Cowperthwaite - Zwisler(雅各布 - 科波威特 - 茨威斯勒)理论，第三种理论也是一种Lennard - Jones - Devonshire理论，只是在混合模型上用Monte - Carlo(蒙特 - 卡洛)方法拟合。Ficket和Davis正确地指出，第一种方法中Kistiakowsky - Wilson状态方程的缺点是压力随温度变化有物理上不合理的行为，并且状态方程中的参数需要用爆轰测量的结果来校准。

对第二种方法 Lennard - Jones - Devonshire 理论的评论是，其优点是可以用非高压实验预测其中的参数，而存在的缺点是在低密度范围下与实际符合不够好，为此 Jacobs 在 1969 年做过改进。事实上早在 1955 年，作者就已经在这篇短小的论文中对上述意见有所评论了。第一是，从物理基础扎实的 Lennard - Jones - Devonshire 理论出发来构筑状态方程，避免了要用高压实验（爆轰）本身来拟合其经验参数的方法。第二是，状态方程的物理行为，包括压力依赖于温度和压力依赖于密度的行为，在大范围内与事实符合。作者在 1962 年开始在中国科学院力学研究所内指导的“高压气体性质的研究”课题，也是循着这一路径方法开展研究的，以期达到对炸药爆轰行为的理论预测。这里给出的是作者根据 Wentorf（温托夫）等人对 Lennard - Jones - Devonshire 理论的表格推导，以及归纳成半经验状态方程的过程和计算用的作图表示。

If the molecules are assumed to be spheres of diameter D, then

$$b = \frac{2\pi}{3} D^3 \qquad (2)$$

1

Thermodynamic Properties of Gas at High Temperatures and Pressures

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1. Equation of State of Dense Gas

When the density of gas is high, it is well known that the simple equation of state for a perfect gas can no longer expected to be valid. The most crude approximation to the equation of state for a dense gas is that of Van der Waal. If P is the pressure, v the volume per molecule, T the temperature, and k the Boltzmann constant, then the Van der Waals' equation is

$$\left(P + \frac{a}{v^2}\right)(v - b) = kT \qquad (1)$$

where a and b are two constants, small in magnitude. The constant b is usually simply identified as four times the volume of a molecule.

At high temperatures, the density of gas can be large only if the pressure is very high. Then term a/v^2 is not important in comparison with P, and Eq. (1) can be simplified into the so-called co-volume equation of state

$$P(v - b) = kT$$

Or we can write

$$\frac{Pv}{kT} = 1 + \frac{1}{\frac{3}{2\pi}\frac{v}{v^*} - 1} \qquad (3)$$

where v^* is a volume defined by $\quad v^* = D^3 \qquad (4)$

However, because of the crude approximation in the Van der Waals' equation of state, neither Eq. (1) nor Eq. (3) can be expected to be sufficiently accurate for gas at very high temperatures and high pressures such as gaseous products of detonation of condensed explosives.

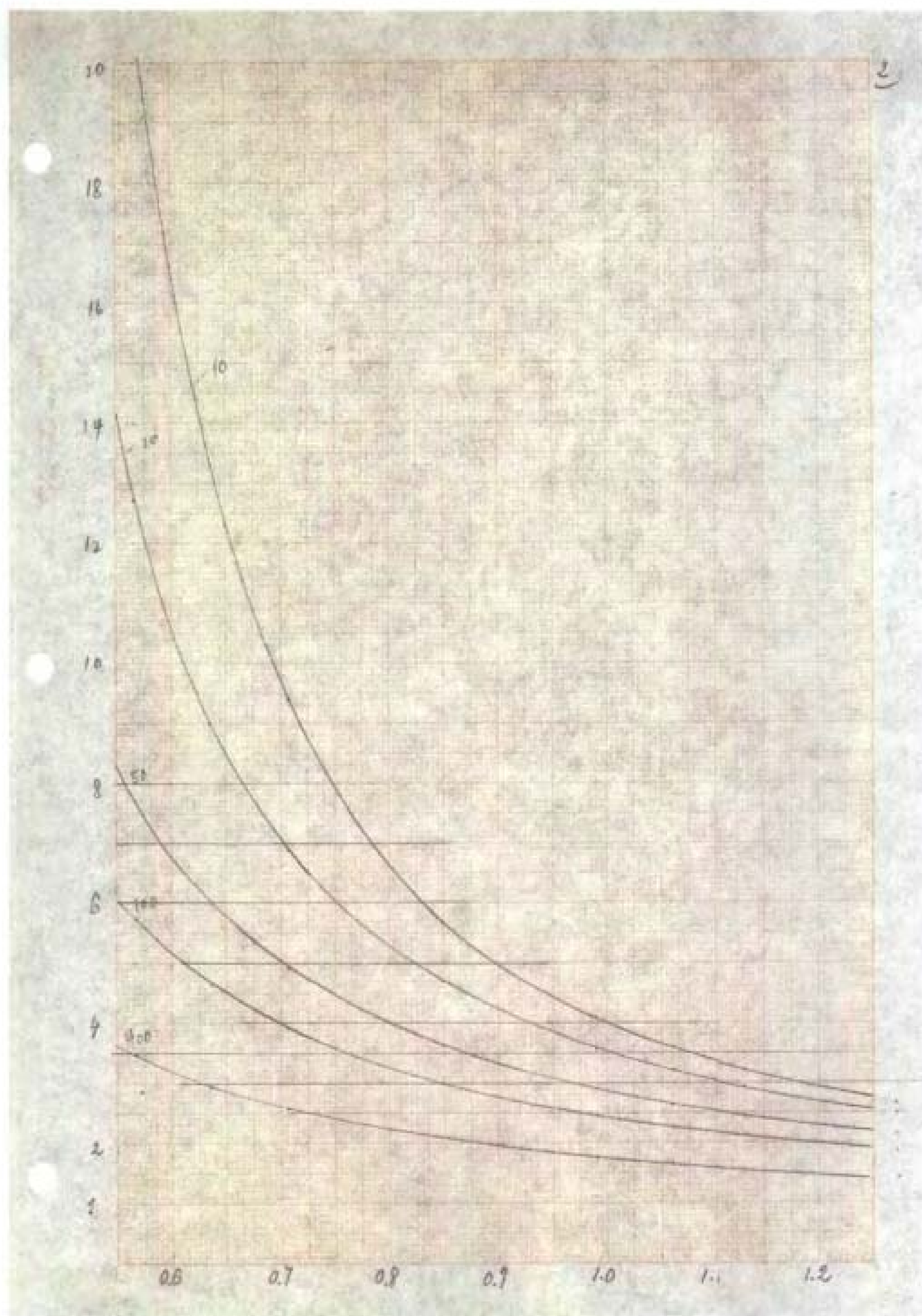
1

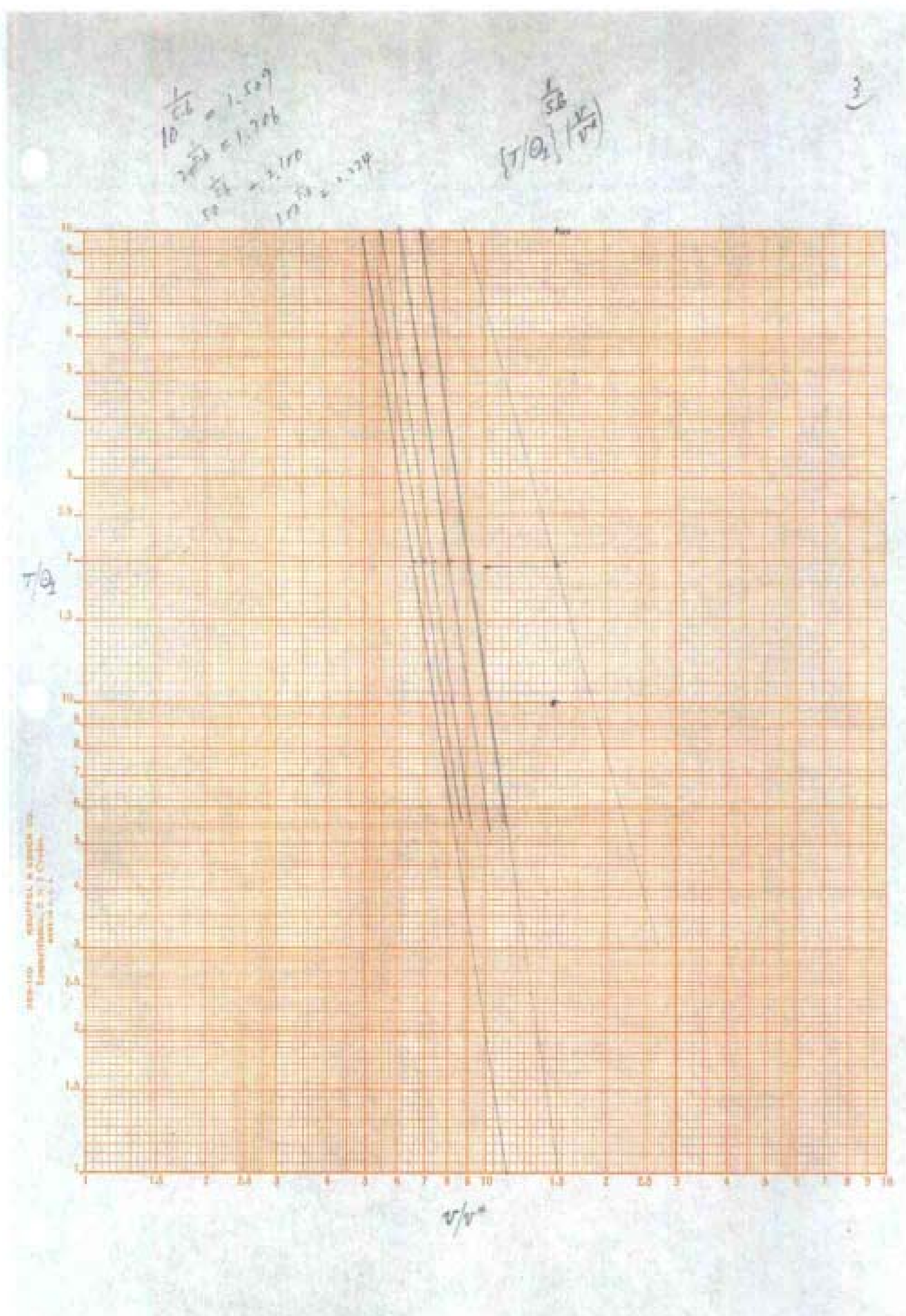
Lennard-Jones & Devonshire Theory for Dense Gas

Calculation by Wentorf, Buehler, Hirschfelder. J. Chem. Phys. 18: 1493 (1950)

Values of $\frac{pv}{kT} - 1$

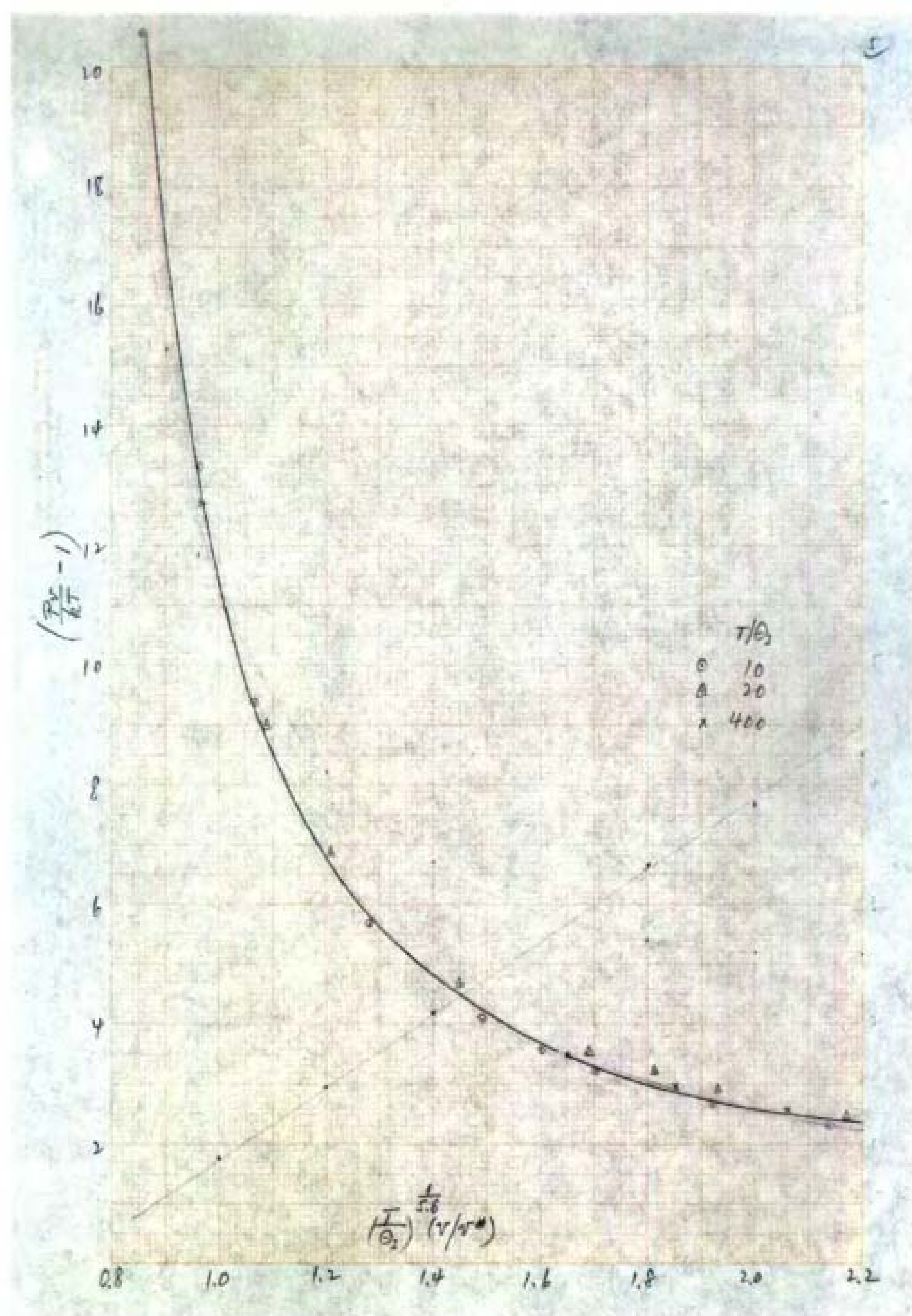
T/T* \ v/v*	0.32	0.405	0.5	0.72	0.98	1.125	1.28	1.62	2
[illegible]	0.5657	0.6364	0.7071	0.8485	0.9899	1.0607	1.1314	1.2728	1.4142
5	35.676	31.471	13.809	6.963	4.393	3.705	3.253	2.595	2.185
7	27.074	16.897	11.312	6.381	4.257	3.720	3.283	2.684	2.390
10	20.584	12.207	9.373	5.659	4.087	3.591	3.242	2.673	2.305
20	13.735	8.978	6.866	4.688	3.576	3.211	2.920	2.486	2.175
50	7.717	5.969	4.910	3.674	2.905	2.652	2.444	2.122	1.885
100	5.733	4.630	3.902	3.000	2.462	2.267	2.104	1.849	1.657
400	3.476	2.962	2.592	2.076	1.777	1.655	1.543	1.319	1.082

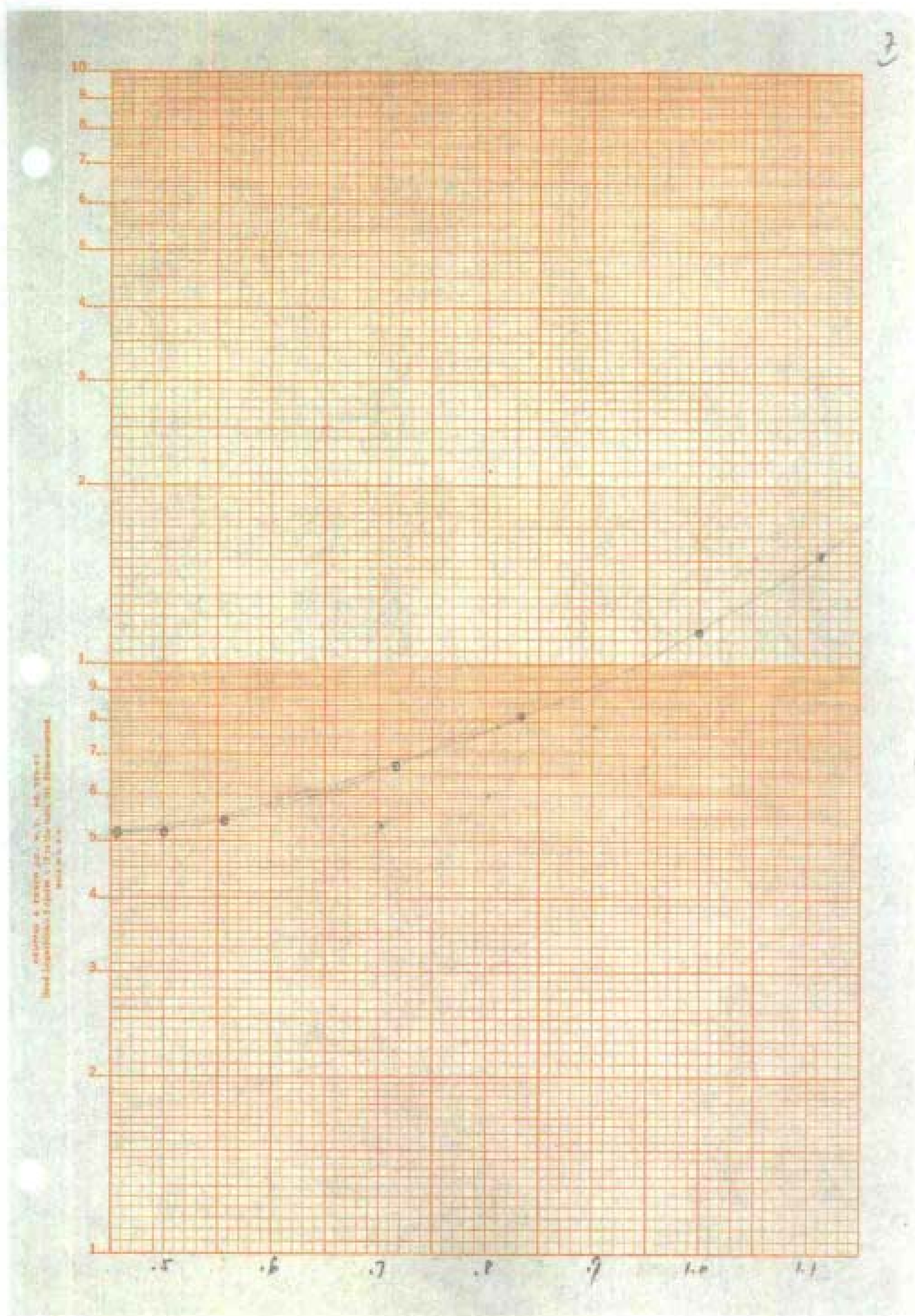

2
20
18
16
14
12
10
8
6
4
2
1
10
50
100
0.6
0.7
0.8
0.9
1.0
1.1
1.2



$$\left(\frac{T}{\Theta_s}\right)^{\frac{1}{5.8}}\left(\frac{v^2}{v^{*2}}\right)$$

T/Θ_s \ v/v^*	0.5657	0.6364	0.7071	0.8485	0.9899	1.0607	1.1314	1.2728	1.4142
10	0.854	0.960	1.066	1.280	1.493	1.600	1.708	1.920	2.136
20	0.966	1.086	1.207	1.449	1.690	1.811	1.931	2.172	2.414
50									
100									
400	1.650	1.853	2.060	2.466	2.886	3.092	3.298	3.706	4.12


5
$\left(\frac{Pv}{RT}-1\right)$
T/Θ_2
○ 10
△ 20
× 400
$\left(\frac{T}{\Theta_2}\right)^{\frac{1}{5.6}}(v/v^*)$
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2
4
6
8
10
12
14
16
18
20



8

Plot $\overset{y}{\dfrac{1}{\frac{Pv}{kT}-1}}$ against $\underset{x}{\left(\frac{T}{\Theta_2}\right)^{\frac{1}{6}}\left(\frac{v}{v_0}\right)}$

$T/\Theta_2 = 10$, $(T/\Theta_2)^{\frac{1}{6}} = 1.468$

x	y
0.832	0.0486
0.935	0.0751
1.038	0.1068
1.246	0.1766
1.453	0.2442
1.559	0.2784
1.661	0.3114
1.870	0.3740
2.076	0.434
2.286	0.491
2.490	0.547
2.592	0.574

$T/\Theta_2 = 20$, $(T/\Theta_2)^{1/6} = 1.647$

x	y
0.933	0.0785
1.049	0.1111
1.164	0.1456
1.399	0.2124
1.630	0.2796
1.750	0.3112
1.865	0.3424
2.100	0.4025
2.230	0.460
2.562	0.515
2.798	0.568

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$T/\Theta_2 = 50, \quad (T/\Theta_2)^{\frac{1}{6}} = 1.920$

x	y
1.067	0.1296
1.221	0.1675
1.357	0.2036
1.630	0.2760
1.900	0.3440
2.040	0.3770
2.172	0.409
2.444	0.471
2.712	0.531

$T/\Theta_2 = 100, \quad (T/\Theta_2)^{\frac{1}{6}} = 2.154$

x	y
1.220	0.1743
1.370	0.2160
1.522	0.2562
1.828	0.3333
2.132	0.406
2.288	0.441
2.440	0.475
2.744	0.541

$T/\Theta_2 = 400, \quad (T/\Theta_2)^{1/6} = 2.714$

x	y
1.526	0.2296
1.727	0.3372
1.917	0.3860
2.204	0.477
2.688	0.562
2.882	0.604

$y = ax - b$

555
177
378

$0.045 = 0.8a - b$

$0.6 = 2.8a - b$

$0.555 = 2a$

$a = 0.278$

$b = 0.8 \times 0.278 - 0.045$

$= 0.222 - 0.045 = 0.177$

$y = 0.278x - 0.177$

$$\frac{Pv}{kT} = 1 + \frac{1}{0.278\left\{\left(\frac{T}{\Theta_2}\right)^{1/6}\left(\frac{v}{\beta}\right)\right\} - 0.177}$$

11

$$\frac{Pv}{kT} = 1 + \frac{1}{0.278\left[\left(\frac{T}{\Theta}\right)^{1/6}\left(\frac{v}{v^*}\right)\right] - 0.177}$$

$$\left(\frac{\partial E}{\partial v}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_v - P$$

$$P = \frac{kT}{v}\left[1 + \frac{1}{0.278\left\{\left(\frac{T}{\Theta}\right)^{1/6}\left(\frac{v}{v^*}\right)\right\} - 0.177}\right]$$

$$\left(\frac{\partial E}{\partial v}\right)_T = -\frac{kT}{v}\,\frac{\frac{1}{6}\,0.278\left\{\left(\frac{T}{\Theta}\right)^{1/6}\left(\frac{v}{v^*}\right)\right\}}{\left[0.278\left\{\left(\frac{T}{\Theta}\right)^{1/6}\left(\frac{v}{v^*}\right)\right\} - 0.177\right]^2}$$

.4275
.177
.2505

$$(E - E_\infty)_T = -\int_v^\infty \left(\frac{\partial E}{\partial v}\right)_T dv = +\frac{kT}{6}\int_v^\infty \frac{0.278\left(\frac{T}{\Theta}\right)^{1/6} d\left(\frac{v}{v^*}\right)}{[\qquad]^2}$$

$$\boxed{(E - E_\infty)_T = \frac{kT}{6}\,\frac{1}{\left[0.278\left(\frac{T}{\Theta}\right)^{1/6}\left(\frac{v}{v^*}\right) - 0.177\right]}}$$

(267)

$$\frac{(E - E_\infty)_T}{(-E^*)} = \frac{1}{6}\,\frac{(T/\Theta)}{0.278\left(\frac{T}{\Theta}\right)^{1/6}\left(\frac{v}{v^*}\right) - 0.177}$$

$$\frac{4.0}{6[0.278 \times 1.526 - 0.177]}$$

$$(c_v - c_{v\infty}) = \frac{k}{6}\,\frac{1}{[\qquad]} - \frac{k}{36}\,\frac{0.278\left(\frac{T}{\Theta}\right)^{1/6}\left(\frac{v}{v^*}\right)}{[\qquad]^2}$$

$$\boxed{\frac{(c_v - c_{v\infty})}{k} = \frac{1}{6}\,\frac{0.2316\left(\frac{T}{\Theta}\right)^{1/6}\left(\frac{v}{v^*}\right) - 0.177}{\left[0.278\left(\frac{T}{\Theta}\right)^{1/6}\left(\frac{v}{v^*}\right) - 0.177\right]^2}}$$

.477
2562
.177
1772

$$(H - H_\infty)_T = (E - E_\infty)_T + (Pv - kT)$$

$$= \frac{kT}{6}\,\frac{1}{[\ ]} + kT\,\frac{1}{[\ ]} = \boxed{kT\,\frac{7/6}{0.278\left(\frac{T}{\Theta}\right)^{1/6}\left(\frac{v}{v^*}\right) - 0.177} = (H - H_\infty)_T}$$

12

$$\boxed{(c_p - c_{p_m})_T = \gamma (c_v - c_{v_m})}$$

We have $\dfrac{\partial\left(\frac{F}{T}\right)}{\partial T} = -\dfrac{E}{T^2}$

Thus $\dfrac{\partial}{\partial T}\left\{\dfrac{(F-F_m)}{T}\right\} = -\dfrac{E-E_m}{T^2}$

$$F - F_m = \frac{Tk}{6}\int_T^\infty \frac{dT'}{\left\{0.278\left(\frac{T'}{\Theta_2}\right)^{1/6}\left(\frac{v}{v^*}\right) - 0.177\right\}T'}$$

$$= Tk\int_{(T/\Theta_2)^{1/6}}^\infty \frac{d\xi}{\xi\left[0.278\left(\frac{v}{v^*}\right)\xi - 0.177\right]} \qquad \xi^6 = T'$$

$$= Tk\int_{0.278\left(\frac{v}{v^*}\right)(T/\Theta_2)^{1/6}}^\infty \frac{d\eta}{\eta\left[\eta - 0.177\right]} = \frac{Tk}{0.177}\int_{0.278\left(\frac{v}{v^*}\right)(T/\Theta_2)^{1/6}}^\infty \left[\frac{1}{\eta - 0.177} - \frac{1}{\eta}\right]d\eta$$

$$= \frac{Tk}{0.177}\log\frac{0.278\left(\frac{v}{v^*}\right)(T/\Theta_2)^{1/6}}{0.278\left(\frac{v}{v^*}\right)(T/\Theta_2)^{1/6} - 0.177}$$

$$\frac{(S-S_m)_T}{k} = -\frac{1}{0.177}\log\frac{0.278\left(\frac{v}{v^*}\right)(T/\Theta_2)^{1/6}}{0.278\left(\frac{v}{v^*}\right)(T/\Theta_2)^{1/6} - 0.177}$$

$$-\frac{1}{0.177}\left[\frac{1}{6} - \frac{\frac{1}{6}\,0.278\left(\frac{v}{v^*}\right)\left(\frac{T}{\Theta_2}\right)^{1/6}}{0.278\left(\frac{v}{v^*}\right)(T/\Theta_2)^{1/6} - 0.177}\right]$$

$$= -\frac{1}{0.177}\log\frac{-}{-} + \frac{1}{6}\frac{1}{0.278\left(\frac{v}{v^*}\right)(T/\Theta_2)^{1/6} - 0.177}$$

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$$\frac{S-S_a}{k} = \frac{1}{6}\frac{1}{0.278\,(T/\Theta)^{1/2}\left(\frac{v}{v^*}\right)-0.177} - \frac{1}{0.177}\log\left\{\frac{0.278\,(T/\Theta)^{1/2}\left(\frac{v}{v^*}\right)}{0.278\,(T/\Theta)^{1/2}\left(\frac{v}{v^*}\right)-0.177}\right\}$$

4.3

Asymptotic Analysis of Some Integrals Connected with Calculation of Spectral Line Absorption Coefficient

关于谱线吸收系数的某些积分的计算

这个工作是1952年初进行的。现存文稿包括：(1) S. S. Penner（潘纳）在1952年1月8日写给钱学森的便笺，要求找到一个计算谱线吸收系数的积分方法，(2) 钱学森为此用两种不同的数学方法进行的推导，共12页。这里只选印了其中一种方法的推导手稿，共5页。

谱线吸收系数的计算在定量光谱学上占有重要地位。定量光谱学是50年代末期被提出来的光谱学中的一个新领域，其标志之一是在1961年出现了一个新的学术刊物《Journal of Quantitative Spectroscopy and Radiative Transfer》（简称JQSRT）。其刊物主编就是在加州理工学院喷气推进中心（由讲座教授钱学森所领导）工作，并与钱学森合作过的S. S. Penner。钱学森回国后回顾那段历史时说到，Penner到喷气推进实验室最初的任务是对喷气推进发动机燃烧过程用光谱的办法进行实验探测。为了这个目的，必须把光谱学的物理原理能够推进到可以定量化的程度，于是开展了包括对光谱吸收系数、发射率和辐射传输问题的研究。钱学森在他的《物理力学讲义》一书中的第13章“热辐射”中，反映了当时的基本考虑。在回国后他曾对“工程光谱学”的专门方向进行过倡导。钱学森认为光谱学要达到工程化，除了物理基础的部分之外，要在定量化的计算上下功夫。这里所给出的是他应Penner的要求对同时有Döppler（多普勒）效应变宽，自然宽度和碰撞变宽条件下的谱线吸收线型的计算问题中的一个积分进行求解。原文未发表过。不过后来Penner在他的专著《Quantitative Molecular Spectroscopy and Gas Emissivities》（定量光谱学和气体辐射），Pergamon (1959) 中，将其结果收入，并指明这一结果是由钱学森首先得到的。实际

上 Penner 书中关于谱线吸收线型的计算引用钱学森的计算结果共有三处，其中只有公式(4.53)之推导在现存手稿中。线型的计算看似简单，实际上难度很大。这些计算工作曾有许多人的努力，其中包括 M. Born（玻恩）的贡献。由钱学森给出的方法是简洁而有力的。他的手稿推导清晰易懂，可见其学术风格非同一般。这一计算的实际背景可参见上述 Penner 专著的第四章。

2

$$I(a;\omega^2) = a\int_1^\infty e^{-a^2\tau^2+\frac{\omega^2}{\tau^2}}\,d\tau$$

$$I(a;\omega^2) = a\int_1^\infty e^{-a^2\left(\tau^2-\frac{k^2}{\tau^2}\right)}\,d\tau$$

where $k = \frac{\omega}{a}$

Let $x = a^2$

$$I(x;\omega^2) = \sqrt{x}\int_1^\infty e^{-x\left(\tau^2-\frac{k^2}{\tau^2}\right)}\,d\tau$$

Let $i(p;k) = \int_0^\infty e^{-px} I(x,k)\,dx$

Then

$$i(p;k) = \int_1^\infty d\tau \int_0^\infty \sqrt{x}\, e^{-x\left(\tau^2+p-\frac{k^2}{\tau^2}\right)}\,dx$$

$$= \Gamma\left(\tfrac{3}{2}\right)\int_1^\infty \frac{d\tau}{\left(\tau^2+p-\frac{k^2}{\tau^2}\right)^{3/2}}$$

$$= \tfrac{1}{2}\Gamma\left(\tfrac{3}{2}\right)\int_0^1 \frac{d\xi}{\left(1+p\xi-k^2\xi^2\right)^{3/2}}$$

$$= \tfrac{1}{2}\Gamma\left(\tfrac{3}{2}\right)\left[-\frac{2(p-2k^2\xi)}{(p^2+4k^2)\sqrt{1+p\xi-k^2\xi^2}}\right]_0^1$$

$$= \Gamma\left(\tfrac{3}{2}\right)\left[\frac{p}{p^2+4k^2} - \frac{p-2k^2}{(p^2+4k^2)\sqrt{(1-k^2)+p}}\right]$$

Inversion, $\frac{p}{p^2+4k^2} \to \cos 2kx$

2

$$\frac{p-2k^2}{p^2+4k^2} = \frac{p-2k^2}{(p+2ik)(p-2ik)}$$

$$= \frac{1}{2}\left(\frac{1}{p-2ik} + \frac{1}{p+2ik}\right) + \frac{ik}{2}\left(\frac{1}{p-2ik} - \frac{1}{p+2ik}\right)$$

$$= \frac{1}{2}\left(\frac{1+ik}{p-2ik} + \frac{1-ik}{p+2ik}\right)$$

$$-\frac{p^2-2k^2}{(p^2+4k^2)\sqrt{p+(1-k^2)}} = -\frac{1}{2}\left[\frac{1+ik}{(p-2ik)\sqrt{p+(1-k^2)}} + \frac{1-ik}{(p+2ik)\sqrt{p+(1-k^2)}}\right]$$

(See Churchill, p. 297)

$$\rightarrow -\frac{1}{2}\left[\frac{1+ik}{\sqrt{1+i^2k^2+2ik}} e^{2ikx} \operatorname{erf}\left(\sqrt{1+i^2k^2+2ik}\,\sqrt{x}\right)\right.$$

$$\left. + \frac{1-ik}{\sqrt{1+i^2k^2-2ik}} e^{-2ikx} \operatorname{erf}\left(\sqrt{1+i^2k^2-2ik}\,\sqrt{x}\right)\right]$$

$$= -\frac{1}{2}\left[\frac{1}{1+ik} e^{2ika^2} \operatorname{erf}\left((1+ik)a\right) + \frac{1}{1-ik} e^{-2ika^2} \operatorname{erf}\left((1-ik)a\right)\right]$$

$$= -\text{Real part of } \frac{1}{1+ik} e^{2ika^2} \operatorname{erf}\left((1+ik)a\right)^{*}$$

$$= -\mathcal{R}\ \frac{\cos 2ka^2 + i\sin 2ka^2}{1+ik} \operatorname{erf}\left((1+ik)a\right)$$

$$= -\mathcal{R}\ \frac{(\cos 2ka^2 + k\sin 2ka^2) + i(\sin 2ka^2 - k\cos 2ka^2)}{1+k^2} \operatorname{erf}\left((1+ik)a\right)$$

---

* erf of complex arguments is discussed by Salzer in April 1951 issue of "Mathematical Tables and Other Aids to Computation"

3

Case 1). $a \gg 1$, and k any value.

$$\frac{1}{1+ik}\, e^{2ika^2}\, \operatorname{erf}\left((1+ik)a\right)$$

$$\cong \frac{1}{1+ik}\, e^{2ika^2}\left[1 - \frac{e^{-(1+ik)^2a^2}}{\sqrt{\pi}\,(1+ik)a}\sum_{n=0}^{\infty}(-1)^n \underbrace{\left(\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{2n-1}{2}\right)}_{n \text{ factors}} \frac{1}{(1+ik)^{2n}a^{2n}}\right.$$

If we put $k = \frac{\mu}{a} = \cot\beta$,

$$1+ik = \frac{1}{\sin\beta}\left(\sin\beta + i\cos\beta\right)$$

$$= \frac{i}{\sin\beta}\left(\cos\beta - i\sin\beta\right) = \frac{i}{\sin\beta}\, e^{-i\beta}$$

$$= \frac{e^{i\left(\frac{\pi}{2}-\beta\right)}}{\sin\beta}$$

$$\frac{(-1)^n}{(1+ik)^{2(n+1)}} = (-1)^n \sin^{2(n+1)}\beta\; e^{-i\left[(n+1)\pi - 2(n+1)\beta\right]}$$

$$= -\sin^{2(n+1)}\beta\; e^{2(n+1)i\beta}$$

$$= -\sin^{2(n+1)}\beta\left[\cos 2(n+1)\beta + i\sin 2(n+1)\beta\right]$$

$$\mathcal{R}\,\frac{1}{1+ik}\, e^{2ika^2}\operatorname{erf}\left((1-ik)a\right) = \frac{1}{1+k^2}\left(\cos 2ka^2 + k\sin 2ka^2\right)$$

$$+ \frac{1}{\sqrt{\pi}}\,\frac{e^{(k^2-1)a^2}}{a}\sum_{n=0}^{\infty}\underbrace{\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{(2n-1)}{2}}_{n \text{ factors}}\;\frac{1}{a^{2n}}\sin^{2(n+1)}\beta\,\cos 2(n+1)\beta$$

4

$$I = \frac{\sqrt{\pi}}{2}\left[\cos 2ka^2 - \frac{\cos 2ka^2 + k\sin 2ka^2}{1+k^2}\right.$$

$$\left. - \frac{1}{\sqrt{\pi}}\frac{e^{\omega^2-a^2}}{a}\sum_{n=0}^{\infty}\underbrace{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{(2n-1)}{2}}_{n\ \text{factors}}\frac{1}{A^{2n}}\sin^{2(n+1)}\theta\cos 2(n+1)\theta\right]$$

$$I = \frac{\sqrt{\pi}}{2}\frac{k}{1+k^2}\left\{k\cos 2ka^2 + \sin 2ka^2\right\}$$

$$- \frac{e^{\omega^2-a^2}}{2a}\sum_{n=0}^{\infty}\underbrace{\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{(2n-1)}{2}}_{n\ \text{factors}}\frac{\sin^{2(n+1)}\theta\cos 2(n+1)\theta}{A^{2n}}$$

Case II), $\omega \gg 1$, and k any value.

$$\frac{1}{1+ik}e^{2ika^2}\operatorname{erf}\left(\left(\frac{1}{k}+i\right)\omega\right)$$

$$\cong \frac{1}{1+ik}e^{2ika^2}\left[1 - \frac{e^{-(1+ik)^2a^2}}{\sqrt{\pi}(1+ik)a}\sum_{n=0}^{\infty}(-1)^n\underbrace{\left(\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-1}{2}\right)}_{n\ \text{factors}}\frac{1}{\left(\frac{1}{k}+i\right)^{2n}\omega^{2n}}\right]$$

$$\frac{1}{k}+i = \tan\theta + i = \frac{1}{\cos\theta}(\sin\theta + i\cos\theta)$$

$$= \frac{i}{\cos\theta}(\cos\theta - i\sin\theta) = \frac{e^{i(\frac{\pi}{2}-\theta)}}{\cos\theta}$$

$$\frac{1}{(1+ik)^2a} = \frac{\frac{1}{k^2}}{a\left(\frac{1}{k}+i\right)^2} = \frac{A}{\omega^2\left(\frac{1}{k}+i\right)^2}$$

$$\frac{(-1)^n}{\left(\frac{1}{k}+i\right)^{2n+2}} = \cos^{2(n+1)}\theta\, e^{-i[(n+1)\pi + 2(n+1)\theta]}(-1)^n$$

$$= -\cos^{2(n+1)}\theta\, e^{+2(n+1)\theta i}$$

5

χ_{aw}

$$I = \frac{\sqrt{\pi}}{2}\frac{a\omega}{\omega^2+a^2}\left\{\frac{\omega}{a}\cos 2\omega a + \sin 2\omega a\right\}$$

$$-\frac{a e^{\omega^2-a^2}}{2\omega^2}\sum_{n=0}^{\infty}\underbrace{\frac{1}{2}\frac{3}{2}\cdots\frac{(2n-1)}{2}}_{n \text{ factors}}\frac{\cos^{2(n+1)}\theta\cdot\cos 2(n+1)\theta}{\omega^{2n}}$$

for $\omega \gg 1$

$$I = \frac{\sqrt{\pi}}{2}\frac{a\omega}{\omega^2+a^2}\left\{\frac{\omega}{a}\cos 2\omega a + \sin 2\omega a\right\}$$

$$-\frac{e^{\omega^2-a^2}}{2a}\sum_{n=0}^{\infty}\underbrace{\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{(2n-1)}{2}}_{n \text{ factors}}\frac{\sin^{2(n+1)}\theta\,\cos 2(n+1)\theta}{a^{2n}}$$

for $a \gg 1$

$$\tan\theta = \frac{a}{\omega}$$

4.4

Emissivity of Diatomic Gases at Low Pressure

双原子气体在低压下的辐射

这项研究工作最早开始于1950年，作者的手稿完成于1951年。现存文稿包括：计算推导、论文手稿，论文打字稿(10页)，及作为附录由S.S. Penner（潘纳）所写的手稿等。

这里发表的是作者于1951年所写的论文打字稿，共10页，稿上留有S.S. Penner阅读后在文稿中局部处提出的改进意见。Penner对处理低压下双原子分子发射率计算给出的一个近似方法的手写稿，题为“Approximate Emissivity Calculation for CO at Atmospheric Pressure and 300 K”（一氧化碳在大气压和300K条件下发射率的近似计算），这里未收录，讨论的是同一个问题，但Penner给出的是基于实验的一个经验公式。在此之前Penner和他的博士生Ostrander（奥斯特兰德）曾经给出过一个在低气压下，谱线呈完全分立情况下的发射率的计算表格，用的是数值方法。在钱学森这一初稿中提出的方案，是基于对前人发射率计算公式中的Bessel函数做了渐近展开，以及用Euler - Maclaurin方法计算转动量子数的求和，数学方法的简洁有力是非常明显的。这三方面的工作最后综合成为一篇文章“Emission of Radiation from Diatomic Gases. Ⅲ. Numerical Emissivity Calculations for Carbon Monoxide for Low Optical Densities at 300 K and Atmospheric Pressure”（双原子气体分子的发射率. Ⅲ. 在300 K和大气压条件下具有低光学密度的一氧化碳发射率的数值计算），发表在《J. Appl. Phys.》（应用物理学报），Vol. 23, No. 2，256—263 (1952)，署名人为S. S. Penner，M. H. Ostrander, H. S. Tsien。查阅了本手稿并与最后发表之文章对比来看，人们可以发现，虽然这篇论文是三个人的工作，但三个人分别的学术贡献和作用仍然可以体察出来。而钱学森的学术风格鲜明地跃然纸上，这也正是他所提倡的物理力学方法论的一种体现。

ROUGH DRAFT

I.

30 July 1951

EMISSIVITY OF DIATOMIC GASES AT LOW PRESSURES

I. INTRODUCTION

Emissivity calculations for diatomic gases from spectroscopic data were developed recently by S. S. Penner (Ref. 1). His method is based upon the use of an average absorption coefficient for the entire fundamental and higher vibration-rotation bands. The method is thus effective when there are extensive overlapping and broadening of the spectral lines, and hence is accurate for gases at high total pressures and temperatures. At low pressures, the lines do not overlap and a different approach to the problem should be made. Penner and M. H. Ostrander (Ref. 2) have computed the emissivity of carbon monoxide for the case of non-overlapping lines by a numerical procedure, using spectroscopic data obtained recently by Penner and D. Weber (Ref. 3). The results are in excellent agreement with the emissivity determined experimentally by W. Ullrich and H. C. Hottel (Ref. 4). The amount of numerical work involved is, however, rather heavy. It is the purpose of the present paper to develope an approximate but convenient formula for calculating the emissivity of diatomic gases for the case of non-overlapping lines.

II. FORMULATION OF THE PROBLEM

If T is the temperature, θ the characteristic temperature, ν the wave number, ν^* the characteristic wave number, P_ν the spectral absorption coefficient at ν , p the partial pressure of the emitting gas,

Notation

ω: wave number
ν frequency
$\theta = h\nu^*/k$

and L the optical path length, then the emissivity ε of the gas under the specified conditions is

$$\varepsilon = \int_0^\infty \frac{\nu^3\{1 - e^{-P_\nu pL}\}\,d\nu}{e^{\frac{hc\nu}{kT}} - 1} \Big/ \int_0^\infty \frac{\nu^3\,d\nu}{e^{\frac{hc\nu}{kT}} - 1} \tag{1}$$

at low temps. when the contributions from upper harmonics is negligibly small

If only the fundamental vibration-rotational band is considered, the absorption coefficient P_ν is given approximately by

$$P_\nu = \frac{b}{\pi}\sum_{j=1}^{\infty}\left[\frac{S^{0\to1}_{j\to j-1}}{(\nu-\nu^{0\to1}_{j\to j-1})^2+b^2} + \frac{S^{0\to1}_{j-1\to j}}{(\nu-\nu^{0\to1}_{j-1\to j})^2+b^2}\right] \tag{2}$$

where b the [illegible] half-width of the spectral lines, and Ss' are the integrated absorptions for the lines centering on the wave numbers corresponding to the indicated transitions. The Ss' can be computed in turn by using the results of J. R. Oppenheimer (Ref. 5). As

$$S^{0\to1}_{j\to j-1} = \frac{N_T \epsilon^2 \pi}{3\mu c^2 Q}\,\frac{\nu^{0\to1}_{j\to j-1}}{\nu^*}\, j\, e^{-\frac{E_{0,j}}{kT}}\, F\cdot G \tag{3}$$

and

$$S^{0\to1}_{j-1\to j} = \frac{N_T \epsilon^2 \pi}{3\mu c^2 Q}\,\frac{\nu^{0\to1}_{j-1\to j}}{\nu^*}\, j\, e^{-\frac{E_{0,j-1}}{kT}}\, F'\cdot G' \tag{4}$$

This factor should take the complete internal partition function for reference to zero as reference state then [illegible] should be multiplied by $e^{-\theta/2T}$ [illegible]

where N_T is the number of emitting molecules at temperature T per unit volume per unit pressure, ϵ the effective charge, μ the reduced mass, c the velocity of light, Q the complete internal partition function, E's are the internal energy levels given by

$$E_{v,j} = k\theta\left[v - xv(v-1) + \gamma j(j+1)\{1 - 4\gamma^2 j(j+1) - \delta v\}\right] \tag{5}$$

$E(v,j) - E(0,0)$

$x = x^*$

$\gamma \simeq B_e/\omega_e = \frac{1}{2}(D_e/B_e)^{1/2}$ } Note approximations

$\delta \simeq D_e/B_e$

$\theta = hc\omega^*/k = h\nu^*/k$

(5)

where x, γ , δ are molecular constants in their standard notations. These constants are non-dimensional and are small. The F's and G's are

$$F(j,\gamma) = 1+4\gamma j\left(1+\frac{5\gamma j}{8}-\frac{3\gamma}{8}\right)$$

$$F'(j,\gamma) = F(-j,\gamma) = 1-4\gamma j\left(1-\frac{5\gamma j}{8}-\frac{3\gamma}{8}\right) \tag{6}$$

and

$$G = 1-\exp\left\{-\left(\frac{hc}{kT}\right)\nu^{0\to 1}_{j\to j-1}\right\}$$

$$G' = 1-\exp\left\{-\left(\frac{hc}{kT}\right)\nu^{0\to 1}_{j-1\to j}\right\} \tag{7}$$

The complete internal partition function can be written as

$$Q = \frac{1}{\gamma\frac{\theta}{T}\left(1-e^{-\theta/T}\right)}\left[1+\gamma\left(\frac{1}{3}\frac{\theta}{T}+\delta\frac{T}{\theta}\right)+\frac{\delta}{e^{\theta/T}-1}+\frac{2x\frac{\theta}{T}}{\left(e^{\theta/T}-1\right)^2}\right] \tag{8}$$

If the fundamental vibration-rotation band gives the main contribution to the emissivity of the gas, the above equations give the necessary information to calculate approximately the emissivity ε .

III. APPROXIMATE SOLUTION

The numerical work is carrying out the computation indicated in the

previous section is very heavy. A short formula, however, can be developed: First of all, when the lines are separated from each other, each line can be considered alone, independent of others. Furthermore, the value of the factor outside of the bracket in the numerator of Eq. (4) can be approximated by its value at the center of each line. Thus according to S. S. Penner (Ref. 6)

$$\varepsilon = \frac{15}{\pi^4}\left(\frac{\theta}{T}\right)^4 \sum_{j=1}^{\infty}\left[\frac{\left(\nu^{v\to v'}_{j\to j-1}/\nu^*\right)^3}{e^{\left(\frac{\theta}{T}\right)\left(\nu^{v\to v'}_{j\to j-1}/\nu^*\right)}-1}\int_{-\infty}^{\infty}\left(1-e^{-P^{v\to v'}_{j\to j-1}\rho L}\right)d\left(\frac{\nu}{\nu^*}\right)\right.$$

$$\left.+\frac{\left(\nu^{v\to v'}_{j\to j+1}/\nu^*\right)^3}{e^{\left(\frac{\theta}{T}\right)\left(\nu^{v\to v'}_{j\to j+1}/\nu^*\right)}-1}\int_{-\infty}^{\infty}\left(1-e^{-P^{v\to v'}_{j\to j+1}\rho L}\right)d\left(\frac{\nu}{\nu^*}\right)\right] \tag{9}$$

where the Ps are the absorption coefficient due to the particular line with transitions as indicated. The integrals can be easily evaluated (Ref. 6) and are given by the modified Bessel functions I_0 and I_1:

$$\int_{-\infty}^{\infty}\left(1-e^{-P^{v\to v'}_{j\to j-1}\rho L}\right)d\left(\frac{\nu}{\nu^*}\right)=2\pi\left(\frac{b}{\nu^*}\right)\xi_j e^{-\xi_j}\left[I_0(\xi_j)+I_1(\xi_j)\right] \tag{10}$$

and

$$\int_{-\infty}^{\infty}\left(1-e^{-P^{v\to v'}_{j\to j+1}\rho L}\right)d\left(\frac{\nu}{\nu^*}\right)=2\pi\left(\frac{b}{\nu^*}\right)\eta_j e^{-\eta_j}\left[I_0(\eta_j)+I_1(\eta_j)\right] \tag{11}$$

where

$$\xi_j = S_{j \to j-1}^{o \to 1} pL / 2\pi b \tag{12}$$

and

$$\eta_j = S_{j-1 \to j}^{o \to 1} pL / 2\pi b \tag{13}$$

A further approximation can now be made. The magnitude of ξ's and η's are generally quite large if the product pL of pressure and optical path length is of the order of unity. Therefore, the asymptotic values of the Bessel functions can be used. Then

$$\int_{-\infty}^{\infty} \left(1 - e^{-P_{j \to j-1}^{o \to 1} pL}\right) d\left(\frac{\nu}{\nu^*}\right) \cong 2\sqrt{\frac{b\, S_{j \to j-1}^{o \to 1} pL}{\nu^{*2}}} \tag{14}$$

and

$$\int_{-\infty}^{\infty} \left(1 - e^{-P_{j-1 \to j}^{o \to 1} pL}\right) d\left(\frac{\nu}{\nu^*}\right) \cong 2\sqrt{\frac{b\, S_{j-1 \to j}^{o \to 1} pL}{\nu^{*2}}} \tag{15}$$

By substituting Eqs. (14) and (15) into (9), the emissivity is calculated as a sum over j.

To carry out the sum over j, one can use the Euler-Maclaurin summation formula (Ref. 7), which evaluates the sum by an integral. First, due to the smallness of γ, δ, the following expansions, including terms up to the square of γj, are approximate.

$$\frac{\nu_{j \to j-1}^{o \to 1}}{\nu^*} = 1 - 2\gamma j - \left(\frac{\delta}{\gamma}\right)\gamma^2 j^2 \tag{16}$$

$(2-\delta)\gamma j$ $\quad 16 j^2 \gamma^2$

$\delta \gamma j \sim .1 \left(\frac{\delta}{\gamma}\right) \gamma^2 j^2$ for $j = 10$

6

$$\sqrt{F} = 1 + 4\gamma j - 3\gamma^2 j^2 \tag{17}$$

$$\frac{\sqrt{G}}{e^{\left(\frac{h}{kT}\right)\left(\nu^{j\to j-1}_{j\to j-1}(\rho')\right)}-1} = e^{-\left(\frac{h}{kT}\right)\left(\nu^{j\to j-1}_{j\to j-1}(\rho')\right)}\left[1-e^{-\left(\frac{h}{kT}\right)\left(\nu^{j\to j-1}_{j\to j-1}(\rho')\right)}\right]^{-2} \tag{18}$$

[illegible handwritten marginal note: "... is ... of the ... formula ... related to ... level)"]

$$\ldots = \frac{e^{-\frac{\theta}{T}}}{\left(1-e^{-\frac{\theta}{T}}\right)^{2}}\left[1+\left(\frac{\theta}{T}\right)\left\{2+\frac{e^{-\frac{\theta}{T}}}{\left(1-e^{-\frac{\theta}{T}}\right)}\right\}\gamma j + \left\{\left(\frac{\theta}{T}\right)\left(\frac{\epsilon}{\gamma}+2\frac{\theta}{T}\right)+\frac{\frac{\theta}{T}e^{-\frac{\theta}{T}}}{\left(1-e^{-\frac{\theta}{T}}\right)}\left(3\frac{\theta}{T}+\frac{\epsilon}{2\gamma}\right)+\frac{3}{2}\frac{\left(\frac{\theta}{T}\right)^{2}e^{-2\frac{\theta}{T}}}{\left(1-e^{-\frac{\theta}{T}}\right)^{2}}\right\}\gamma^2 j^2\right] \tag{19}$$

and

$$\frac{[E(0,j)-E(0,0)]}{kT}$$

$$e^{-\frac{E_{vj}}{kT}} = e^{-\frac{\gamma}{2}\frac{\theta}{T}j^{2}}\left[1-\frac{1}{2}\left(\frac{\theta}{T}\right)\gamma j+\frac{1}{8}\left(\frac{\theta}{T}\right)^{2}\gamma^{2}j^{2}\right] \tag{20}$$

The corresponding quantities for the transitions $j - 1 \rightarrow j$ can be very easily obtained from Eqs. (16) to (20) by replacing j with $-j$. Because of this property of symmetry, the sum of terms from the transition $j \rightarrow j - 1$ and the transition $j - 1 \rightarrow j$ for every j is a function of j^2 only. Thus, after appropriate canceling of linear terms,

$$\frac{\left(\nu^{j\to j-1}_{j\to j-1}(\rho')\right)^{2}}{e^{\left(\frac{h}{kT}\right)\nu^{j\to j-1}_{j\to j-1}(\rho')}-1}\int_{-\infty}^{\infty}\left(1-e^{-P^{j\to j-1}_{j\to j-1}(L)}\right)d\left(\frac{\rho'}{\rho}\right)+\frac{\left(\nu^{j-1\to j}_{j-1\to j}(\rho')\right)^{2}}{e^{\left(\frac{h}{kT}\right)\nu^{j-1\to j}_{j-1\to j}(\rho')}-1}\int_{-\infty}^{\infty}\left(1-e^{-P^{j-1\to j}_{j-1\to j}(L)}\right)d\left(\frac{\rho'}{\rho}\right)$$

$$= \phi\left(\frac{\theta}{T}\right)e^{-\left(\frac{\theta}{T}\right)}f\left(\gamma,\epsilon,\frac{\theta}{T}\right)\sqrt{\left(\frac{2\epsilon}{\rho}\right)\left(\frac{Ab}{\rho^{2}}\right)}\,\nu_j^{2}\,e^{-\frac{\gamma}{2}\frac{\theta}{T}j^{2}}\left[1+g\left(\gamma,\epsilon,\frac{\theta}{T}\right)\gamma^{2}j^{2}\right] \tag{21}$$

where

$$A = \frac{N_r \epsilon^2 \pi}{3 \mu c^2} \frac{T}{\theta} \tag{21}$$

A is thus a constant independent of temperature and pressure. The f function is simply deduced from the partition function Q as given by Eq. (8):

$$f = 1 - \gamma\left(\frac{1}{6}\frac{t}{T} + 4\frac{T}{t}\right) - \frac{\delta/2}{(e^{t/T}-1)} - \frac{x\,\frac{t}{T}}{(e^{t/T}-1)^2} \tag{22}$$

Therefore f is a quantity close to unity. The function g is computed from the expansions given in Eqs. (16) to (19). It is

$$g = \frac{3}{2}\left\{\frac{\left(\frac{t}{T}\right)e^{-t/T}}{1-e^{-t/T}}\right\}^2 + \frac{\left(\frac{t}{T}\right)e^{-t/T}}{1-e^{-t/T}}\left\{\frac{5}{2}\frac{t}{T} + \ldots\right\} + \ldots \text{[illegible]} \tag{23}$$

The Euler-Maclaurin formula can be now employed to evaluate the sum in the emissivity $\mathcal{E}$. The resulting integral over j extends from 1 to ∞. But this range can be made to be from 0 to ∞ by simply deducting the approximate value of the integral from 0 to 1 from the extended integral. Thus

$$2\sum_{j=1}^{\infty} j\, e^{-\frac{\theta}{2T}j^2}\left[1 + g\left(\ldots\right)\ldots\right] \cong 2\int_1^{\infty} j\, e^{-\frac{\theta}{2T}j^2}\left[1 + g\gamma j^2\right]dj + \frac{1}{12}$$

$$\cong 2\int_0^{\infty} j\, e^{-\frac{\theta}{2T}j^2}\left[1 + g\gamma j^2\right]dj - \frac{5}{12}$$

$$= \Gamma\left(\tfrac{3}{4}\right)\left(\frac{2T}{\gamma\theta}\right)^{3/2}\left[1 + \frac{3}{2}\frac{\gamma T}{\theta}g\right] - \frac{5}{12}$$

The $\Gamma\left(\frac{3}{4}\right)$ has the numerical value of 1.225. Finally then the expression

The emissivity for the case of non-overlapping lines is

$$\varepsilon = \frac{30}{\pi^4}\left(\frac{\theta}{T}\right)^5 e^{-\theta/T} f\left(\gamma,\delta,x,\frac{\theta}{T}\right)\left[\Gamma\left(\frac{3}{4}\right)\left(\frac{2T}{\gamma\theta}\right)^{3/4}\left\{1+\frac{3}{2}\frac{\sqrt{T}}{\theta}g\left(\gamma,\delta,\frac{\theta}{T}\right)\right\}-\frac{5}{12}\right]\sqrt{\left(\frac{\gamma b}{\nu^*}\right)\left(\frac{ApL}{\nu^*}\right)} \qquad (24)$$

where f and g are functions given previously in Eqs. (23) and (24).

Since the value of f is nearly unity and the factor before g in Eq. (24) is small, a good approximate equation for the emissivity is

$$\varepsilon \cong \frac{30}{\pi^4}\left(\frac{\theta}{T}\right)^5 e^{-\theta/T}\,\Gamma\left(\frac{3}{4}\right)\left(\frac{2T}{\gamma\theta}\right)^{3/4}\sqrt{\left(\frac{\gamma b}{\nu^*}\right)\left(\frac{ApL}{\nu^*}\right)} \qquad (25)$$

IV. APPLICATION TO CARBON MONOXIDE

For carbon monoxide, the molecular constants are

$$\theta = 3066.9^\circ K$$
$$\nu^* = 2142.3\ cm^{-1}$$
$$\gamma = 0.000895$$
$$\delta = 0.0091$$
$$x = 0.00620$$

The value of A computed from the measurements of Penner and Weber (Ref. 3) is

$$A = 22.95\ atm^{-1}\ cm^{-2}$$

They have also determined (Ref. 8) b to be 0.077 cm^{-1} at one atmosphere of total pressure.

According to the approximate equation (25), the emissivity at T = 300°K and a total pressure of one atmosphere

$1.62 \times 10^{-3} \sqrt{pL}$

$$\mathcal{E} = 1.597 \times 10^{-3} \sqrt{pL} \qquad (26)$$

where pL is in atm-cm. By using the more exact equation (25), the emissivity is

$$\mathcal{E} = 1.606 \times 10^{-3} \sqrt{pL} \qquad (27)$$

1.640

The difference between the approximate value and the more exact value is quite small. The comparison between the computed emissivity and the measurements of Ullrich and Hottel (Ref. 4) is shown, in Fig. 1. The agreement is quite satisfactory up to pL of approximately 10.

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工 程 科 学

5.1

Engineering and Engineering Sciences
"工程和工程科学"的报告提纲

作者于 1936 年从师于美国加州理工学院的力学大师 Theodore von Kármán（冯·卡门），1939 年取得博士学位后留校成为 von Kármán 的得力助手，在 von Kármán 的指导和影响下，从事应用力学的研究，围绕高速飞行的"声障"和"热障"以及薄壳结构的稳定性等问题发表了一系列经典文献；接着，他又为火箭技术的发展开创了理论研究。1945 年作者被美国国防部聘任为由 von Kármán 任团长的科学咨询团的团员，为美国空军提供长远发展的意见，同年 5 月随同团长 von Kármán 考察德、英、法等国的航空事业，特别是德国火箭技术的发展情况。

1947 年初，36 岁的钱学森已经成为近代力学、航空和火箭技术优秀的世界一流科学家，并且进入了美国麻省理工学院年轻的正教授的行列。作者已在十余年的研究和教学的丰富实践中，深切领悟了以 Prandtl（普朗特）—von Kármán 为代表的应用力学学派的精神，并熟谙了这一学派的研究手法。作者剖析了第二次世界大战期间工业特别是武器的飞速发展的根源，认识到：在本世纪初，由德国 Göttingen（哥廷根）学派的 Felix Klein（克莱因）所倡导的科学与工程相结合从而推动工业技术发展的思想已经得到了充分的体现；事实上，这种结合和推动已经成为决定国力的强盛和人民生活福利水

平的关键。作者敏锐地认识到在自然科学和工程技术之间已经形成了一个独立的科学体系，那就是工程科学；在20世纪的前40年，应用力学正是这一工程科学的代表，而到了40年代，工程科学面临着火箭技术、电子技术和核技术等在更广阔的工程技术范围内的发展，因而也正面临科学与技术紧密结合的需求和时机；为了促进工程技术和社会经济的发展以及国力的强盛，需要提倡这种工程科学的发展。

1947年夏，作者回国探亲，先后访问了浙江大学、交通大学和清华大学，以“工程和工程科学”为题，就工程科学的内涵和特点、研究内容和方法、当前的研究领域，特别是工程科学在中国发展的重要性等方面做了讲演。

这里选印的是作者在回国之前准备讲演而写的提纲，共有2页。

1

Engineering and Engineering Sciences

Lecture to be delivered at July 28, 1947

1) During the past decade great advancement in engineering has been achieved mainly through the general adoptation of intensive engineering research. Typical examples: a) Aviation; b) Electrical applications; c) ~~Atomic~~ nuclear energy.

2) What is engineering research? a) Developmental research
b) Basic research

3) Felix Klein's concept of relation between science and engineering. — Historical relation between science & engg.

4) What is the characteristics of the ~~[illegible]~~ aim of "Applied Mechanics" "engineering mechanics"

Developmental

b) Preliminary investigation to eliminate unfavorable ~~methods~~ procedures and mode of attack. Examples
i) Rocket fuels
ii) Neutron fuels

c) Preliminary investigation to determine whether ~~[illegible]~~ a proposal is feasible.
i Long range rocket   ii) New Modes of Propulsion.
~~ii) [illegible]~~

~~d)~~ c) To analyze the possible cause of difficulties in engineering project. i) Flutter of suspension of bridge

~~5) Methods.~~

Basic

a) Understanding of phenomena in the most basic sense:
i) Turbulence
ii) Thin shell theory

b) ~~Future~~ Anticipation of future Developments
i) Supersonic flow
ii) Hypersonic flow

2

5) Methods. Not just theory:

a) Close cooperation between theory and experiment < Not only logical method / But intuition
b) Physics, Chemistry & other natural sciences.
c) Higher mathematical Methods
   i) Analysis
   ii) Partial Diff. Equation
   iii) Machine computation

6) The Present Realm of investigation in basic science

a) Fluid mechanics
b) Elasticity — thermal shock
c) Plasticity
d) Combustion
e) Electronics
f) Solid State — material
g) Nuclear energy
h) Biological sciences & engineering application.

7) The Importance of Engineering Science in China.

1) 15
2) 15
3) a) 15
   15
   15
   b) 15
   15

5.2

Engineering and Engineering Sciences

工程和工程科学

这是发表于1948年的"Engineering and Engineering Sciences"（工程和工程科学）一文的手稿，共有24页。这篇论文的内容是以作者在1947年夏天回国时给浙江大学、交通大学和清华大学的学生所做的同名讲演为基础的。作者在讲演中系统地介绍了工程科学的内涵、工程科学家的任务以及作为一名工程科学家需要接受什么样的教育和训练。

作者在引言中，首先回顾了本世纪上半叶科学技术的研究愈益成为决定国家和国际事务中的关键这一震撼人心的历史事实，其中最富戏剧性的实例，乃是第二次世界大战时期雷达和原子弹的发展，对世界民主力量的伟大胜利做出了卓越的贡献。科学和技术的研究不再是无计划的个人活动，任何一个大国的政府都认识到，这种研究实为增强国力和国民福利的关键，因而严密地加以组织和控制，使之成为现代工业不可分割的组成部分。作者意识到纯科学的发现与工业应用之间的距离已经很短，而留长发的纯科学家和理短发的工程师之间的差别也非常之小，他们之间紧密合作的实际需要产生了一个新的职业，那就是工程科学家，他们在纯科学与工程之间架起桥梁，运用基础科学知识解决工程问题。

开创这种工程科学的研究，在历史上可以溯源到20世纪初德国Göttingen哥廷根大学的伟大的数学家F. Klein（克莱因），他所开创和领导的学派中产生了像L. Prandtl（普朗特）、Theodore von Kármán（冯·卡门）和S. Timoshenko（铁木辛柯）等那样杰出的工程科学家。20世纪20年代中，Timoshenko和von Kármán相继移居美国，把这一学派的传统风格带到美国，并通过他们的学生广泛传播到美国的著名大学和研究单位。到了40年代，美国著名的理工院校已经充分认识到理工合一的教育原则的必要性并付诸实施。

作者1947年回国的讲演就是向祖国的著名大学宣传工程科学的重要性以及理工合一的教育原则，充分反映出作者急盼祖国繁荣昌盛的赤子之心。

1949年祖国得到解放，作者归国报效心切，历尽美国政府的阻挠和迫害，1955年终于回到祖国的怀抱。回国以后，作者继续提倡发展工程科学，并且积极展开培养工程科学家的工作。首先，在回国的第二个月里，就受命创建中国科学院力学研究所。作者当时的建所模式不只限于力学，还包括了自动控制、工程经济、物理力学等，实际上是按照工程科学的框架来建所的。1956年起，作者和钱伟长一起创办了三期力学研究班；1958年，作者和郭沫若、严济慈、华罗庚等一起组建了中国科技大学，开始大批培养工程科学家的工作。1957年，作者在《科学通报》上发表了题为“论技术科学”的论文，按国内的习惯将“工程科学”改称为技术科学，论文进一步全面地论述了技术科学的范围、方法论以及培干和组织等各个方面。70年代，哈尔滨军事工程学院迁往长沙，组建国防科技大学（初期称长沙工学院）。时任国防科委副主任的钱学森又将这一理工结合的教育思想贯彻于该校的建校方针之中，再次强调在工科院校中要加强基础理论的教育，使培养出来的学生能适应现代科技的飞速发展。80年代末，钱学森根据他一生从事科学研究和科学管理的切身感受，提出培养“科技帅才”的观点。他认为，为了建设“四个现代化”，在科技队伍的顶层，需要有科技帅才。他们不仅要有雄厚的自然科学理论知识、丰富的工程实践经验，而且要有社会科学和哲学的修养，要文理工相结合。

从他提倡工程科学的1947年到今天，整整半个世纪过去了。历史发展的实践充分地证明了，一个国家的国力和国民福利的强大和技术科学的发达程度之间存在着息息相关的联系，由此看出，作者倡导的发展工程科学或技术科学的思想是多么先进和重要。今天我们重读这篇文章，仍然感受到许多新的启示。

Double Spacing
4 carbons

1

Engineering and Engineering Sciences*

Hsue-shen Tsien**

Introduction

When we review the development of human society in the last half of century, one is, perhaps, struck by the phenomenal growth of the importance of technical and scientific research as a determining factor in national and international affairs. It is quite clear that while technical and scientific research were pursued in an individual and unplanned manner during the earlier days, such research is now carefully planned and controlled in any major nation. Thus technical and scientific research has become a matter of state along with the age-old matters such as the financial policy, or the foreign relations, agricultural development. A closer examination for the reason of this growth of the importance of research would naturally yield the answer that research is now an integral part of modern industry and one cannot speak of a modern industry without mentioning research. Since industry is now the foundation of a nation's strength and welfare, technical and scientific research is then the key to a nation's strength and welfare.

But then one might argue that since the pioneering days of industrial age, technical and scientific research was related to the industrial development, then what is exactly the reason for

* First given in lectures for the engineering students at the National Chekiang University, the Chiao-tung University, and the National Tsing-Hua University during the summer of 1947

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making the research so important to-day? The answer to this question is the rate at which the modern industry is forced to develope due to national and international competition. At this rapid rate of developement, research must be greatly intensified with the almost immediate application of basic scientific findings. Perhaps, nothing is so dramatic as an illustration as the wartime development of radar and nuclear energy. That the successful development of radar and nuclear energy contributed much to the victorious conclusion of the World War II on the side of Democracy is an established fact. Thus here intensive research has brought the findings of the basic science of physics through practical engineering and to successful applications to weapons of war in the short interval of a few years. Thus the distance between a pure scientific fact and industrial application is now very short. In other words, the difference between a long-haired pure scientist and a short-haired practical engineer is very small indeed, and a close cooperation between them is essential for the successful development of the industry.

This need for close cooperation of the pure scientist and the practical engineer produced a new profession — the engineering research men or engineering scientist. They form the bridge between the pure science and the engineering. They are the men who apply the basic scientific knowledge to engineering problems. The purpose of the present article is to

discuss what the engineering scientist can do, what is their job, in engineering. And then what kind of education and training he needs in order to do the job assigned to him.

Contributions of a Engineering Scientist to Engineering Development

The contributions of a engineering scientist to engineering development can be briefly stated as the economy in effort both in manpower and in money. This economy is achieved through a sound and general analysis of the problem on hand to point out 1) whether a proposed engineering scheme is at all feasible 2) if feasible, what would be the best way of carrying out the proposal and finally 3) if a certain project failed, what is the cause of failure and what would be the remedy. It is needless to say that if an engineering scientist can fulfill these assignments then the cut and try in any research and development is to a large extent eliminated. All the effort and money can then be concentrated on the best approach or the few better method of attacking the problem having the best chance of success.

It might be argued, nevertheless, that these three main problems assigned to the engineering scientist are really the three basic problems in engineering. What is that an engineering scientist can do which an engineer cannot do? The answer to this question is that as the engineering profession become more and more complex, there is a need for specialization

the present requirement of knowledge for a satisfactory solution of the three problems stated includes a good training not only in engineering but also in mathematics, physics and chemistry, as will be discussed in greater detail in the subsequent paragraphs. Therefore the training of an engineering scientist is quite different from the conventional training of an engineer. In other words, he must be the specialist to solve just the three basic problems of engineering development.

The Feasibility of a Proposal — Long Range Rockets

To gain a better understanding of the way in which an engineering scientist solves the three basic problems of engineering development, a few illustrative examples will be described. The first example is the investigation of the possibility of extremely long range rocket. A rocket is propelled by the reaction of the exhaust jet obtained through the combustion of the propellant carried. The performance of the rocket motor is expressed as the specific propellant consumption in pounds of propellant per hour required to generate one pound of thrust. This figure is slightly modified by the change of the atmospheric pressure, but can be generally taken to be constant and equal to an average value. The specific propellant consumption thus fixes the motor performance. The range of the rocket will evidently depend upon the amount of propellant carried, i.e., the ratio of the propellant weight to the gross weight of the rocket. This is the propellant loading ratio. The rocket, during its flight, is opposed by the air

resistance. We see thus that for an engineering scientist to solve the problem of the possibilities of long range rocket he has to have three kinds of basic information: the rocket motor performance, the structural efficiency and the aerodynamic forces at high speeds. For rocket motor performance he will depend upon the rocket engineer for test data. For the structural efficiency he will depend upon the stress man for data. For aerodynamic forces at high speeds, he will depend upon the high speed windtunnel testing for data. The engineering scientist on the job must then synthesize these informations by good engineering judgement, by the application of the laws of dynamics and the skill of solving differential equations. The result: the calculated range of the rocket. If he uses the best rocket motor performance, the lowest practical value of specific fuel consumption, if he uses the best construction to achieve the highest propellant loading ratio, and if he uses the best aerodynamic design of the shape of the rocket to minimize the air resistance, he will then arrive the longest range that can be achieved by a rocket.

This formulation of the problem of long range rocket assumes that the best motor performance, best structural efficiency and best aerodynamic shape are known to the analyst. But the real situation may not be so easy. An engineering scientist will find that while previous experience shows that the performance of the propellant can be predicted to within 10% accuracy

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by making a careful calculation of the combustion temperature and the composition of the exhaust gas using chemical and thermodynamic equilibrium in the rocket combustion chamber and by then calculating the exhaust velocity of the rocket using the dynamics of gaseous flow, the types of propellant actually tested ~~is~~ are very few. In searching for the best possible rocket performance, he may wish to know the probable specific propellant consumption for more energetic ~~chemical~~ [illegible] reactions. This means the engineering scientist has make the theoretical estimate of these untried propellant. For instance, he may wish to calculate the performance of the liquid fluorine and liquid hydrogen rocket. ← Insert p. 6a !!!

Similarly, the engineering scientist may find the information on structural efficiency and aerodynamic forces quite incomplete. He is then forced to investigate a particular promising type of structure or to investigate high speed flow over aerodynamic shapes to determine the probable air resistance at high speeds. In other words, to solve the problem of the possibility of long range rocket, the engineering scientist may be required not only to solve a very difficult problem in exterior ballistics but may also have to solve problems in thermodynamics and combustion, in elasticity and strength of materials and in fluid mechanics. His problem is thus not ~~much~~ ~~difficult~~ easy. but his rewards is also large. ~~[illegible] what is possible and what is not possible in the future rocket.~~

If he does this (think of calculation), he will find two important facts about chemical rocket propellants. These are:

1) There is a strong tendency for the ordinary combustion products such as carbon dioxide and water to dissociate at the extremely high temperatures of the combustion chamber and these dissociations absorb heat. Therefore calculations on the propellant performance using low temperature calorimeter data is totally unreliable. In other words, thermodynamics and chemical equilibrium are matters of primary importance here.

2) There is no "wonder" propellant which will give a tenfold increase in the performance, i.e., lower the specific consumption to one tenth of the value for present propellant. This is easily seen from the following table (Ref.1) which shows that the best propellant is the combination fluorine and hydrogen which gives a specific consumption not less than one half of the more conventional nitric acid and aniline combination.

Hence by this kind of investigation, the engineering scientist can achieve a broad orientation in an entire new field of engineering. He knows what to expect and is able to judge critically the validity of the claims made by any inventors. Such ability of judgement generally requires years to achieve, if try and error is the only method used. Engineering science is then the useful tool to shorten this crucial process of "learning the trade".

Best Method of Attack — Manufacture of Fissionable Material

Very often in engineering practice, one is confronted with the problem of choosing the best method of attack among a possible few. Here again the services of the engineering scientist is invaluable. Let us take the example of the manufacture of fissionable material. According to H. D. Smyth (Ref. 2), the different possible methods are

1) Manufacture of plutonium-239 by the slow neutron pile using natural uranium and the chemical separation of plutonium

2) Manufacture of uranium-235 by electromagnetic separation from the [illegible] uranium-238 in the natural uranium

3) Manufacture of uranium-235 by separation from uranium-238 utilizing thermal diffusion

4) Manufacture of uranium-235 by isotope separation utilizing gaseous diffusion

All methods except the first one is a physical process in that the materials to be separated have identical chemical properties. During the development of nuclear energy for the atomic bomb of the United States, all four methods were pursued. This way of attacking all possible methods simultaneously is certainly a wartime expedient when the time is very limited and the success of the project is a dire necessity. In normal times, this mode of proceeding with the project is certainly wasteful as we shall see presently. In normal times engineering scientists should be called into service to

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analyze the four different process to determine which one of them is the most economical. Of course, the engineering scientists will need much detailed informations which must be obtained by either theoretical analysis or experimentations. For instance, in the first method, he would have to determine the fission cross-section, or fission probability, of uranium-235, the resonant absorption cross-section of uranium-238, the scattering and the absorption cross-section of the moderator, etc. Then by the known principles of nuclear physics, he has to work out the process of the diffusion of neutrons in the pile, the distribution of neutron density in the pile, and finally the critical size of the pile. He has also to find the best method of constructing the pile by trying out in his calculation the different ways of placing the pieces of the uranium and the moderator. After all this investigation, the engineering scientist is able to say that the probable economy of the manufacture of the plutonium-239 by the slow neutron pile method.

Now by a similar approach with laboratory experiment and theoretical calculation, the engineering scientist would be able to estimate the economy of all other proposed methods. Then the question of the best method to manufacture the fissionable material can be answered. It is quite evident now that if such an analysis of the relative economy of the different process were possible, the plutonium process, or method 1), will be the chosen method. General (Leslie R.) Groves has revealed to the

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McMahon Committee that as of June 1945, the monthly operating cost of the processes were:

Hanford Plutonium Plant	$ 3,500,000
Oak Ridge Diffusion Plant	6,000,000
" " Electro-magnetic Plant	12,000,000

Thus the Hanford Plutonium Plant is the most economical one in spite of the fact that it must be the one with the largest capacity for the fissionable materials.

Now suppose the preliminary analyses by the engineering scientist decides on the plutonium process, what would be the consequences? According to General Groves' opinion, the investment spent and committed for plants and facilities as of June 30, 1945 is as follows:

Manufacturing facilities:	Hanford Plant	$ 350,000,000	
	Others	892,000,000	
	Total		1,242,000,000
Housing for workers:	Hanford Plant	$ 48,000,000	
	Others	114,500,000	
			162,500,000
Research			186,000,000
Workmen's compensation + medical care			4,500,000
Total			1,595,000,000

Therefore if the authority that directed the development of nuclear energy for the United States during wartime were able to decide

on the flutter ~~having upon the advice of engineering scientist~~, then approximately one billion dollars could have been saved. In other words, two third of the investment could have been saved, if it were possible to use fully the services of engineering scientists.

<u>Reason and Remedy for a Failure — The Tacoma Narrows Bridge</u>

The third problem which may require the attention of an engineering scientist is the discover the reason and method of remedy for ~~a~~ failure. While the two previous problems, the investigation of the feasibility and the best method of attacking a new development, are work to be done before starting the main part of the engineering, the third problem is, of course, something to be done afterwards. Let us take the example of the Tacoma Narrows Bridge. This bridge was first opened to traffic on July 1, 1940 and it was a suspension bridge of extremely narrow roadbed, as can be seen in the following table of dimensions:

Dimensions of the first Tacoma Narrows Bridge, Washington, USA

Center Span	2800 ft.
2 side spans, west side	1100 ft.
" " east side	1100 ft
West side Back Stay	497 ft
East side " "	261.8 ft
Total Length	4759.2 ft.
Width of Roadway	26 ft.
Total Width including sidewalks	39 ft.

After this bridge was built, it was found that the bridge was extremely flexible. During windy nights, a ghost effect often occurred as the head lights of approaching cars appear and then disappear caused by the lateral + longitudinal oscillation of the roadway. On 10:00 AM, November 7, 1940 the bridge started to oscillate rather violently in torsion by a gusting strong wind. This oscillation increased its amplitude and finally an hour later the bridge broke at approximately the mid-span. Of course, there is then developed among civil engineers great interest as to the cause of failure of the bridge, a kind never dreamed before. Civil engineers normally deal with static forces of rather large magnitudes. For instance, the stress in the bridge member is generally of the order of tons of force per square inch. Now the air or wind force on a surface is probably of the order of one fifth of a pound per square inch. It was rather difficult at first for the civil engineers to understand how such small wind forces could have broke the strong bridge.

The true mechanism of failure was finally clearly explained by a Committee composed of O. H. Ammann, T. von Kármán and G. B. Woodruff (Ref 3). The report was a typical example of investigation by engineering scientists. It consisted of model testing and theoretical computation. The true reason for the failure of the bridge was the resonant oscillation excited by the wind forces — a phenomena well-known to aeronautical engineers as the flutter, but quite outside the experience of a

civil engineers. The wind forces although small have the same period or is always in phase with the oscillation of the roadway and therefore can built up the amplitude of oscillation to a dangerous magnitude. It is seen then that by incorporating damping and by stiffening the bridge to increase the natural frequency the failure can be avoided. This is the principle for the design of the new bridge.

Here again, the services of an engineering scientist is able to clear up a most perplexing engineering question, and can be used to avoid further mistakes in an engineering design.

Unification — Basic Research in Engineering Science

From the above discussion, it might be construed that the problems in engineering science are individual problems and the engineering scientist is to deal with particular cases without much generalized scheme. This impression is however not correct. Among the multitude of problems in the current development of engineering there are phenomena which occur repeatedly in many branches of engineering. These phenomena can then be abstracted out the direct routine problems which the engineering scientists has to solve and be formulated into individual fields of research. The results of such research will then not only benefit one field of engineering, but to all of them. This is the basic research in engineering sciences, through which the greatly diversified engineering activities are united.

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Historically, such basic research in engineering sciences was started by the great mathematician F. Klein in the Göttingen University, Germany shortly before the World War I. His school has produced such eminent engineering scientists as Th. von Kármán and S. Timoshenko. At that time, the main fields of engineering activity had to deal with mechanisms. It is then natural that the basic research in engineering science was simply called "applied mechanics" (angewandte Mechanik). However, the ever-widening fields of engineering now extend to subjects which are not treated in the applied mechanics as first conceived by the German school. Let us then divide the (current) topics of basic research in engineering sciences into three categories: 1) Research in the field which are not within the old boundary of applied mechanics; 2) Research in field which are near the old boundary of applied mechanics, and 3) Research in the fields which are within the old boundary of applied mechanics. It will be profitable to examine these fields of research in greater detail:

To understand what is the character of basic research in engineering sciences, its ramifications and relations with different engineering problems.

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fully utilized in planning the future engineering research in the Institute. In the following sections, the possible directions of fundamental engineering research which could be carried out in the spirit of applied mechanics will be briefly stated.

II. Research in the Fields which are not within the Old Boundary of Applied Mechanics

The everwidening fields of engineering now extend to subjects which are not treated in applied mechanics as first conceived by the German school. Three subjects seem to belong to this category: the solid state of matter, the electronics, and the atomic engineering.

a) Solid State of Matter

The engineering science of metallurgy has really progressed very little beyond the application of Gibb's phase rule. In fact, the present knowledge of materials is obtained through a tremendous amount of tedious laboratory tests. This large body of empirical data has practically no coordination or systematic interpretation. On the other hand, the physical theory of solid state, based upon quantum mechanics, is developed generally by physicists as a branch of pure science. In other words, there is a wide gap between the practical engineering and the scientific investigation. This gap has to be bridged. This effort of unitizing the physical theory with metallurgy will bring about not only a systematic interpretation of the accumulated empirical data but certainly will also indicate new possibilities in the field of development of materials. It is also certain that after a satisfactory engineering science of materials is developed, the search for an engineering material to satisfy a given specification will be greatly facilitated.

Another field of investigation is the ceramic materials. The present engineering materials are dominated by metals which consist of atomic crystals. There is no reason to believe that the other types of material, consisting of ionic crystals such as the ceramic materials, cannot be utilized as engineering materials for machine construction. In fact the recent demand for materials to withstand extremely high temperatures naturally points research in this direction.

b) Electronics

The electronics engineering can be divided into two main divisions: The division which deals with electronic tubes themselves and the division which treats the circuits and the radiation fields. The second division mainly involves an application of the classical Maxwell theory. The general character of the results is expected, in spite of the fact that such calculations may be very complicated and may require advance mathematical technique. The performance of tubes is, however, seldom comprehensively analyzed. The design of these tubes is generally worked out by numerous tests, guided by a few basic principles. However, the electronics engineering has now passed its heroic age of invention and creation and has entered the age of engineering development. The empirical approach may not be the most economical one in this new situation where detailed improvement of the various devices has to be carried out. This is especially true for very high frequency tubes where the electron inertia

effect can no longer be neglected. It seems necessary to develop an engineering method of calculating such flow fields of electron cloud under the combined effect of rapidly varying external electric and magnetic fields. If this is done, then the characteristics of electronic tubes or other similar devices can be analyzed and the experimental data coordinated.

c) ~~Atomic~~ Nuclear Engineering

While the understanding of the nuclear structure is yet to be achieved, a general interpretation of the nuclear reaction seems to have developed to a satisfactory degree. In fact, the elementary processes of nuclei reactions such as collision, capture, excitation, and emission of new particles from the compound nuclei could be measured and studied individually. If these empirical data are available, then the overall macroscopic performance of the reaction can be predicted by applying the methods of chemical kinetics. Impossible and undesirable processes can then be eliminated from further consideration and large scale tests. This approach to the utilization of atomic energy, or atomic engineering, seems to be able to lead to fruitful results without the danger of uncontrolled experiments. In other words, the stage seems to be set for the rapid development of utilization of nuclear reactions similar to the utilization of molecular reactions such as combustion.

III. Research in the Fields which are Near the Old Boundary of Applied Mechanics

a) Combustion

The theory of combustion has been studied by chemists mainly from the point of view of chemical kinetics. However, the problems which grow out of the recent development of jet propulsion generally involve very high speeds of flow. In such problems the effect of the inertia of the fluid elements certainly cannot be neglected. In fact, a study of a simple one dimensional problem has shown rather unexpected results due to this inertia effect. Then a complete and satisfactory solution of combustion problems must combine the science of fluid motion, i.e., hydrodynamics with the science of chemical kinetics. As an initial approach to this problem the effect of diffusion and turbulence on combustion must be studied.

b) Metal Forming by Plastic Deformation.

The large number of metal forming processes is based upon the plastic deformation of the material. For instance, the widely used process of sheet metal forming is by pressing. This process, until recently, was practically carried out purely by experience. During the design of the dies for this process one has to use the cut and try method guided by a few empirical principles. This method is generally very uneconomical. It then seems necessary to develop a satisfactory theory so that such dies can be designed for each individual problem without resorting to numerous experiments. This new science of plastic forming, of course, will be based upon the methods of the theory of elasticity and a complete knowledge of the solid state of matter which is another topic of research as stated in the previous section.

II. Research in the fields which are within the Old Boundary of Applied Mechanics

a) Turbulence

During the last 15 years the problem of turbulence in fluid flows has been studied intensively and simple rules have been developed for satisfactory solutions of engineering problems in this field. However, the theory still lacks an explanation of the fundamentally important fact that the exchange coefficient in turbulent flows is enormous, as compared with that of laminar flows. The correct understanding of this phenomenon is the hub of the turbulence problem. It is believed that this understanding can only be achieved through a detailed survey of the turbulent fluid field together with theoretical analysis. Measurements on the turbulent velocities, the correlation coefficients and diffusion characteristics must be carried out.

Another possible field of investigation would be the application of the presently known knowledge of turbulence to the other fields of engineering such as combustion and the mixing problems in chemical engineering. Such applications are believed to be extremely useful ~~but rather simple from the point of view of hydrodynamicists~~.

b) Gasdynamics

The recent advance in aeronautics makes the science of gasdynamics one of the most important and urgently needed knowledges. The fundamental problems are connected with the interaction of viscosity and compressibility of the fluid. It was believed that the effects of viscosity, or Reynolds number, and the effect of compressibility, or Mach number, can be separated. However, it is now realized that such separation is impossible. On the other hand, the problem of the interaction is very complex, particularly due to the possible appearance of turbulence in the fluid. The detailed phenomenon must be studied simultaneously by both theoretical analysis and by experiment. In conjunction with this investigation, the effects of second viscosity coefficient and the relaxation time should be considered.

The possibility of flight at extreme speeds presents another very interesting problem of fluid dynamics at a very high Mach number. It is known that at such high Mach numbers, say Mach number greater than 5, the fluid behaves very similarly to a stream of particles. In other words, the fluid reaction on a moving body will be very similar to that predicted by Newton on the assumption of no interaction between the particles of the fluid. The question of flying at extremely high altitudes leads to another interesting problem. This is the problem of fluid motion at extremely low density, i.e., the fluid flow in which the mean free path of the molecules is comparable to the dimension of the body moving in it. The solution of these problems is believed to be essential for the next assignment of aviation, the trans-oceanic flight at velocities faster than the velocity of sound.

Insert p. 16a !

16a

From the above discussion on the different field of basic research in engineering sciences, it would seem that the subject are well within the general field of physics. But then why should they be called research topic in engineering science? The reasons are twofolds. Firstly, there is a fundamental difference between the point of view of a physicist and the point of view of an engineering scientist. The physicist's point of view is that of a pure scientist, interested essentially in simplifying the problem to such an extent that an "exact" solution can be made. The engineering scientist is more interested to solve the problem as given to him. It will be complicated, so only approximate solutions, though sufficiently accurate for engineering purpose, will be attempted. Thus physicist will give exact solutions of an oversimplified problem while engineering scientist wants the approximate solutions of the realistic problem. The work of a physicist may be at times impractical, but that of an engineering scientist must always be practical. The second reason for separating the engineering science from the general field of physics is simply that physicist have no deep interest in engineering problems. Because of these two reasons, the engineering scientist is forced to pick up where the physicist has left it and develop the physical principles into tools of practical engineering.

Training of a Engineering Scientist

For the engineering scientist to solve the problems assigned to them and to carry out research in the basic engineering science he needs definitely an education quite different from the education of an engineer. Then what is exactly the necessary training of an engineering scientist? It would perhaps be best to first see what is the necessary tools for an engineering scientist, —he must have these tools. These are

1) Principles of engineering design and practice
2) Scientific foundations of engineering
3) Mathematical method in engineering analysis.

In the first group of tools, is the conventional engineering subjects such as mechanical drawing, drafting and machine design, engineering materials and processes, shop practice. In the second group of subjects, is the physics and the chemistry which is generally contained in a good engineering curriculum. But here the training of an engineer scientist would be different from the conventional engineer in that he must know much more about physics and chemistry. For instance, his knowledge about mechanics must not stop at the statics and dynamics of rigid bodies and the stresses in simple beams and columns. He must learn the principles of the theory of elasticity and plasticity. His knowledge about fluid motion must not stop at the hydraulics. He must learn the principles of hydrodynamics and fluid mechanics. His

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knowledge in thermodynamics must not stop at the first law and the second law or calculation of the idealized Otto cycle or Diesel cycle. He must know the physical meaning of entropy from the point of view of statistical mechanics and the broad concept of thermodynamical equilibrium. Then he must know the elementary structure of matter from the nucleus up to molecules. In other words, he must know many subjects which a physicist or a chemist has to learn.

The third group of subjects is mathematical methods and principles of mathematics which would help the understanding of the use of mathematical methods. Thus it includes subjects such as advanced calculus, functions of complex variable, principles of mathematical analysis, ordinary differential equations, partial differential equations. In other words, he must know most of the subjects which an applied mathematician should know.

It is quite evident that the prospect engineering scientist cannot hope to cram all his learning into four years of college. In fact he has to be first trained in general engineering which may take three years in a good engineering school after high school, and then he has to spend approximately another three years to learn the science and the mathematics. Therefore it takes at least six years after high school to train an engineering scientist while the general practice now to train the conventional engineer requires only four years. Engineering

scientists are then definitely specialists which are only a few percent among the total personnel engaged in engineering and industry and which has to be trained from persons having the particular talent and inclination.

However as yet, only the necessary tools of the engineering scientists have been discussed. The fact that he is given these tools does not necessary mean that he is trained in using these tools. How can he be trained to use these tools? Here the degree of training cannot be measured in the number of course the student takes or the number of years he spent in the school. Learning how to use the tools effectively can be only achieved by experiencing the ~~using~~ use of the tools. Of course, the process can be accelerated with the aid of expert guidance. Therefore to complete the training of an engineering scientist after six years of schooling, the prospective ~~man~~ scientist get spent one to two years working on a specific problem under the supervision of an experienced senior man. A good way of realizing this would be to study for a doctor's ~~thesis~~ degree with an authoritative instructor in a ~~good~~ well-equipped university. The unhurried academic atmosphere in an educational institution is certainly conductive to thinking which is, after all, the only way to gain wisdom. Wisdom gives insight to a complex problem and insight to a problem is the key to its successful solution.

Concluding Remarks

The training of a competent engineering scientist is a long process of seven to eight years, and the effort and

ability required to complete such training is also proportionately great. Intimately the reward is also large. — From the discussion on the character of work performed by engineering scientists, it can be seen that they form the nucleus of new men in any engineering development, they are the pioneers of new frontiers in industry. In fact the very essence of engineering science — the technique of transforming a basic scientific truth to practical means of human welfare actually goes beyond the realm of present industry. Medicine is the application of chemistry, physics and physiology to help a person to recover from an illness. Agriculture is the application of chemistry, physics and plant physiology to produce food. Both are then engineering in the broad sense of the word and both will then benefit from the methods of engineering science. Hence it is appropriate to call the engineering scientists the most immediate and direct workers towards the goal of the pursuit of science, "aptly expressed by Professor Harold C. Urey: "We wish to abolish drudgery, discomfort, and want from the lives of men and bring them pleasure, leisure and beauty".

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Rocket Propellants

Calculated Performance at Sea-Level, chamber Pressure 20 atmospheres

Oxidizer	Fuel	Weight ratio Oxidizer/Fuel	Chamber Temperature °Rankine	Specific consumption lbs per hr. per lb thrust
Fluorine	Hydrazine	1.146	6,970	12.33
	Hydrazine	2.371	9,500	11.50
	Hydrogen	18.85	10,210	10.20
	Hydrogen	9.42	8,530	9.71
	Hydrogen	6.28	8,296	10.20
Oxygen	Ethyl Alcohol (75% + Water 25%)	1.375	5,530	15.45
	Gasoline	2.62	5,930	14.95
	Hydrogen	2.40	5,500	10.20
Red Fuming Nitric Acid	Aniline	3.000	5,525	16.30

其　他

6.1
化学流体力学

6.1.1
Gas Turbine Cycle for the Manufacture of Nitric Oxide
用燃气透平制造一氧化氮

作者从喷气推进技术的研究中得到启示，认为航空发动机实为一高效的化学反应器，因为它具有体积小、反应快、冷却快、因而效率高，而且可以精确地加以控制等优点，可以运用航空发动机所依据的气体动力学原理，设计制造出高效的化工反应器。作者为了探讨上述设想的可行性，选择了一种比较简单的情况，即利用燃气透平来生产一氧化氮。为此，作者考虑了几种可能的热力学循环过程，并进行了估算。作者又考虑到一氧化氮的产率与燃烧所得到的高温之间有着密切的联系，因而提出了一个求取最优温度的问题。

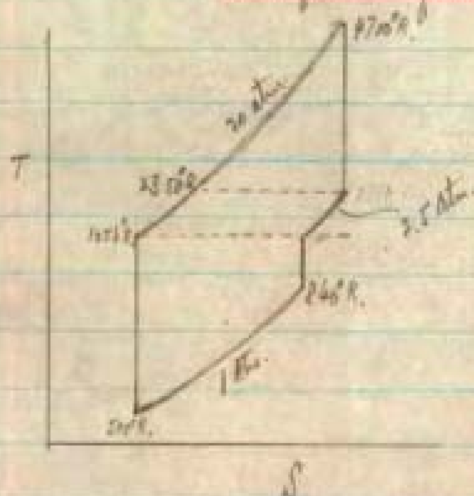
为了进行上述分析，作者还收集了多种气体的化学平衡常数和热力学性质的数据表。

由此可以看到，作者早年的专业特长虽属航空和火箭导弹等国防科技领域，但他那时就注意到这些尖端科技有可能转为民用，从而推动民用工业以至整个国民经济的发展。作者后来在广泛的科技经济领域发表了许多精辟的见解是与那时的思想一脉相承的。

这里选印作者未发表的3页分析部分的手稿，工作时间不详，估计在1953年以前。

1

Gas Turbine Cycle for the Manufacture of Nitric Oxide



1. Ideal cycle with constant $\gamma = 1.333 = \frac{4}{3}$

$$\frac{\gamma}{\gamma-1} = \frac{4}{1} = 4.$$

a) Air temp = 500°R. Comp. ratio = 20.

Comp. temp = $500\,(20)^{1/4}$ = 1056°R. 20 atm.

Heated to 4700°R. by exhaust from high pressure turbine and combustion.

Exhaust from high pressure turbine = 2350°R.

Pressure = 20/8 = 2.50 atm. ← MACH NO. OF NOZZLE $= \sqrt{6} = 2.45$

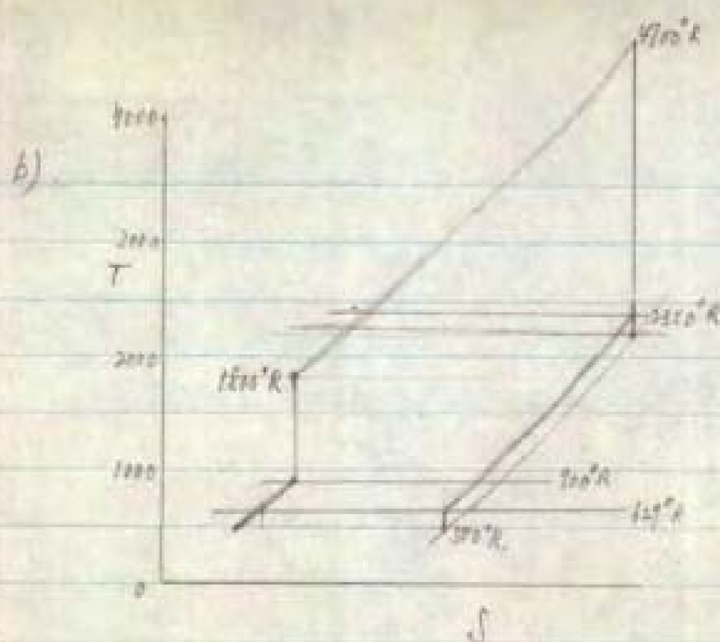
Exhaust cooled by comp. air to 1056°R.

Exp. to atm. = 840°R. = 380°F.

$$\frac{460}{380}$$

2

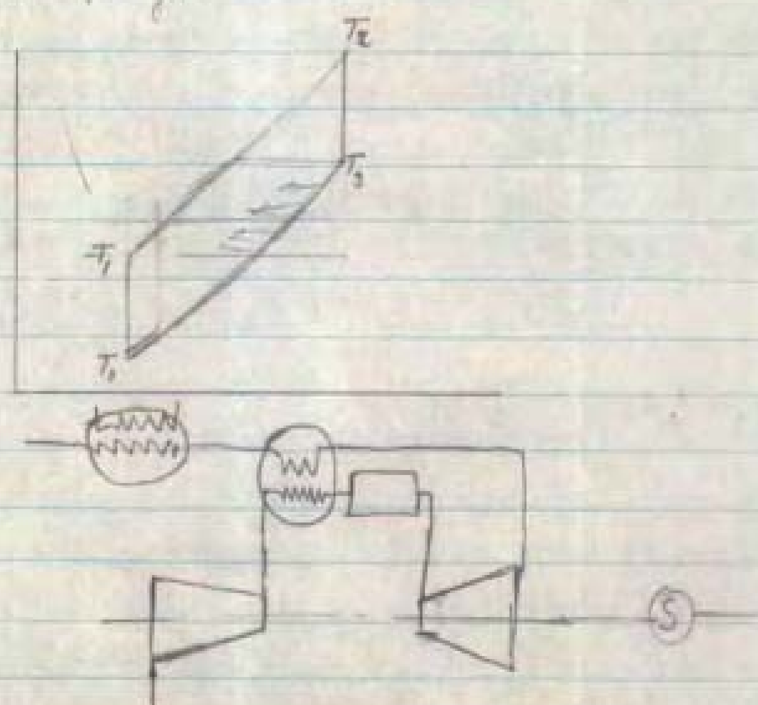
b)



c) High Pressure Turbine exhaust temp = 1800°R.

Pressure ratio $\left(\frac{4700}{1800}\right)^4 = \underline{46.5}$

d) Use simple cycle



2

2. Optimum Problem

If T_3 = chosen temp to give satisfactory purging,
then T_2 and T_3 also gives the pressure ratio.

The heating from T_2 to T_2 by combustion. If T_2 is increased, more fuel is burned and less O_2 for NO, but the formation of NO is favored by higher T_2. So an optimum T_2 exist.

3.

6.1.2

On the Possibility of Manufacturing Chemicals by Gas Dynamical Processes

用气体动力学过程制造化工产品的可能性

1953 年 6 月，作者写成题为 " On the Possibility of Manufacturing Chemicals by Gas Dynamical Processes " （用气体动力学过程制造化工产品的可能性）的手稿，但未发表。

如果说在上面一篇手稿中作者为了想说明有可能利用气体动力学原理来生产化工产品，而只是对用燃气透平制造一氧化氮的可能性作了一个估算，那么这一篇手稿则更为完整地阐述了利用气体动力学原理生产化工产品的可能性。

作者认为当时研究超声速气体动力学的理论和实验的水平已经相当高，不仅清楚了解了高速气流的性质，而且可以为指定的工程目的而实现对高速气流的控制。因此，可以运用气体动力学原理为反应气体创造高温和高压的反应条件，使之实现快速的反应而生成所需的化合物，然后，为了不让化合物在缓慢的冷却过程中进行逆反应，而让介质通过迅速的膨胀和冷却，把已生成的所需化合物"冻结"下来，成为产品。作者介绍了两种可能的途径，一种就是上面一篇手稿中所设想的利用透平机的压缩和膨胀过程，作者称之为气体透平过程；另一种则采用先是击波后接膨胀波的过程，作者称之为气波机过程，造成比前一种过程更为迅速的造成高温高压然后转为冷却冻结的过程。作者认为气波机过程比气体透平过程更迅速和更有效，而且实现的可能性更大，并且建议，研究的第一步是做击波管实验，以便为确定优化条件和设计制造流程取得必需的反应数据。

在手稿的末尾，作者将 S. S. Penner（潘纳） 的题为 " Shock - Tube Experiments for the Production of Chemical Compounds " （生产化合物的击波管实验）的短文用作附录（文中的少量修改是钱写的）。从 Penner 的短文

中可以清楚看出，首先建议用气体动力学途径生产化工产品的正是钱学森，想法来自于钱学森对喷气推进和物理力学的研究。作者回国建立了中国科学院力学研究所后，成立了化学流体力学的研究组，后来又扩大成立了相应的研究室。

这里选印了他的手稿，连同作为附录的 Penner 的短文，共有 15 页，包括：正文 11 页，文字完整连贯，而页码次序有所疏漏；以及附录 4 页。

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On the Possibility of Manufacturing Chemicals by Gas-Dynamical Processes

H. S. Tsien
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(June, 1953)

Dr. S. S. Penner has previously submitted a brief memorandum, here included as Appendix –, on the shock-tube experiments for the production of chemical compounds. The purpose of this note is to expand and to generalize this thesis and to indicate the possibility of manufacturing chemicals by gasdynamical process.

1. Recent Progress in Gasdynamics

Gasdynamics is a branch of fluid mechanics treating the motion of gaseous medium at high speeds. It is a field of engineering science developed intensively during the last twenty years in connection with aeronautical engineering. Recently gasdynamics has received fresh impetus from the exigency of supersonic flight. As a result of the intensive research, the nature of high speed gas flows is now clearly understood for flow speeds up to many times of the sound speed; and even more important, today we know how to control the gaseous flows at such speeds for any specified engineering purpose.

The requirements of supersonic flight also led to the development of special power plants for propulsion. These power plants are rocket, ramjet, turbojet and their various combinations. They are characterized by the very high temperature occurring in their operation. New materials and new cooling methods were devised to cope with these high temperatures. Therefore high temperature is no longer an insurmountable problem in engineering practice.

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2. Effect of Temperature and Pressure on the Chemical Equilibrium and the Rate of Approach to Equilibrium

The equilibrium concentration of a product of chemical reaction will be increased by raising the temperature and pressure if the reaction is endothermic and if the reaction leads to a decrease in the total number of moles of the compounds. Therefore if such a chemical reaction produces a desired chemical compound, the high temperature and pressure will favor the production of the chemical.

A mixture of molecular species is, however, not necessarily at its chemical equilibrium, because the approach to equilibrium takes finite time. The rate of approach to equilibrium may be so small and the time to the equilibrium state may be so long that the mixture can be kept at non-equilibrium state for a very long interval. This low rate of reaction can be always realized by making the temperature of the mixture low. In fact the reaction rate decreases very rapidly with decreasing temperature.

3. Principle of Manufacturing Chemicals by Gasdynamical Processes

With the effect of temperature and pressure on the chemical reaction in mind, the principle of chemical production favored by high temperature and high pressure can be formulated in very simple terms: The reaction must be first carried out at high temperature and high pressure not only to increase the equilibrium concentration of the desired product, but also to shorten the reaction time so that the equilibrium will be reached in the shortest possible time. The hot reaction product at high pressure should then be chilled very rapidly so that the mixture is brought to a low enough temperature where the approach to the new unfavorable equilibrium is slow. Thus the desired chemical is "frozen" at its concentration corresponding to the equilibrium at high temperature and high pressure, and can be

separated out by the conventional processing methods.

How can the high temperature and high pressure be achieved by gasdynamical processes? They can certainly be achieved by isentropic compression. If p_1 is the pressure before compression, p_2 the pressure after compression, T_1 the temperature before compression and T_2 the temperature after compression, then

$$\left(\frac{T_2}{T_1}\right) = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} \tag{1}$$

where γ is the ratio of the specific heats. Equation (1) is obtained with the assumption of perfect gas and constant specific heats. This relation is plotted in Fig. 1 for various values of γ. It is seen that large temperature ratio can attained if the pressure ratio is high. For instance, with $\gamma = 1.2$, for heating from 600°K. to 2,400°K, the pressure ratio required is 407.

Conversely, rapid cooling can be accomplished by expansion if the expansion can carried out rapidly. This can be done by expansion through a De Laval nozzle. If the subscript 3 denotes quantities before expansion and the subscript 4 denotes quantities after expansion, and if M_4 is the Mach number, the ratio of speed of gas at the exit of the nozzle to the speed of sound in the gas at the condition of the exit, then

$$1 + \frac{\gamma-1}{2} M_4^2 = \frac{T_3}{T_4} = \left(\frac{p_3}{p_4}\right)^{\frac{\gamma-1}{\gamma}} \tag{2}$$

Thus if the temperature ratio is 4, and $\gamma = 1.3$, then the exit Mach number is $M_4 = 4.46$. Hence supersonic speeds are definitely involved. Since the flow speed at the entrance to the De Laval-nozzle is very small, the average flow speed may be taken to be around 1,500 ft./sec. Then if the length of the nozzle is 4 inches or 1/3 foot, the time for passage of the gas is $1/(3 \times 1500)$ sec. During this time, the gas is chilled from say 2400°K to 600°K., the rate of chilling is thus

$$\frac{2\gamma}{\gamma+1} = \frac{2.6}{2.3} = 1.13043$$

$$\frac{\gamma-1}{\gamma+1} = \frac{0.3}{2.3} = 0.13043$$

M_1	M_1^2	M_1^2-1	$\frac{p_2}{p_1}=1+\frac{2\gamma}{\gamma+1}(M_1^2-1)$	$1+\frac{\gamma-1}{\gamma+1}(M_1^2-1)$	$\frac{T_2}{T_1}$		
1	1	0	1	1	1		
2	4	3	4.391	1.391	1.527		
3	9	8	10.043	2.043	2.280		
4	16	15	17.956	2.956	3.217		
5	25	24	28.130	4.130	4.647		
6	36	35	40.565	5.565	6.271		
7	49	48	55.261	7.261	8.189		

8.10×10^6 °K/sec. This is indeed very rapid cooling.

Compression can be also achieved by the shock wave, which is a pressure wave of extremely small thickness travelling at supersonic speed with respect to the gas at rest before the shock. If M_1 is the Mach number of the shock wave, and if the subscripts 1 and 2 for quantities before the shock and after the shock, then

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1) \qquad (3)$$

and

$$\frac{T_2}{T_1} = \left[1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1)\right]\left[1 + \frac{\gamma-1}{\gamma+1}(M_1^2 - 1)\right] \Big/ M_1^2 \qquad (4)$$

These relations are plotted in Figs. 2 and 3, (for $\gamma = 1.3$) and are based again on the approximating assumption of perfect gas and constant specific heats. It is seen that high pressure and high temperature can be attained if the shock Mach number is large.

Conversely, cooling can be obtained by an expansion wave, where the isentropic relation of Eq. (1) holds. Expansion waves are thicker than the shock wave, but their thickness are generally much less than the length of a De Laval nozzle. Therefore even higher cooling rates than that obtainable with De Laval nozzle are possible.

4. Gas Turbine Process

How can the proposed gasdynamical process of chemical production be carried out? Generally speaking, there are two types of processes: One type of process uses the steady compression and expansion in a turbo-compressor and gas turbine combination, and may be designated as the gas turbine process. The other type of process uses the compression and expansion generated by shock wave and expansion wave, and may be designated as

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the wave engine process. Both are continuous processes.

Consider the gas turbine process first and take the specific example of the manufacture of nitric oxide, NO, from air.* A schematic diagram of this system is presented in Fig. 4. Air is first compressed by turbocompressors to, say, 20 atmospheres. The compressed hot air is further heated by absorbing heat from the turbine exhaust in a heat exchanger. The heated compressed air, at a temperature, say, 1300°K, is burned with the addition of hydrocarbon fuel in the combustion chamber. With optimum amount of fuel, the concentration of nitric oxide in the combustion product is somewhat greater than 1%. The temperature of the combustion gas is 2500°K. By expanding the combustion gas through the gas turbine, the heat energy of the combustion gas is extracted and transformed into mechanical energy which is utilized to drive the compressors and, if there is a surplus, to drive, say, an electric generator. Since it is of primary importance for the NO production to have very rapid chilling, the first stage of the gas turbine should be of the 'impulse' type so that the temperature of the gas can be immediately reduced to a low value, say, 1600°K, to freeze the nitric oxide concentration at about 1%. Thus the pressure ratio of the first impulse stage of the turbine should be approximately 8. The pressure of the gas after the first stage is then 2.5 atmospheres. The remainder of the stages of the turbine can be of the 'more efficient' reaction type. The turbine exhaust is further cooled by passing through the heat exchanger. The result is a relatively cool gas containing

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* This problem has been analyzed in considerable detail by Gordon J. MacLeod. The study is available as a thesis (California Institute of Technology, Mechanical Engineer degree, 1951), entitled "Some Considerations in the Application of a Gas Turbine Cycle to the Manufacture of Nitric Oxide".

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approximately 1% NO (at ground level).

Present day aircraft turbojet engines have a air flow rate as high as 300 lb per sec. A gas turbine of comparative size can thus process 13,000 tons of air per day. At 1% nitric oxide, this machine can produce 130 tons of NO per day, or 60 tons of fixed nitrogen per day. An important byproduct of the gas turbine process is the surplus power generated. For a machine of the size indicated, the surplus power may be as high as 15,000 kw.

The essential differences between the indicated gas turbine process for NO production and the conventional gas turbine power plant are

1) High temperature of the combustion product
2) Impulse stages of high pressure ratio for the gas turbine.

The conventional gas turbine fails to produce nitric oxide in any appreciable concentration simply because the lack of these characteristics. Both of these design features, however, introduce engineering problems such as cooling of the turbine nozzle and turbine blades. However these problems do not seem to be insurmountable in light of the present knowledge of high temperature design and high speed flows.

5. Wave Engine Process

To use shock waves and expansion waves, the process can be carried out a simple device called shock tube. This versatile device has already been used by Drs. N. Davidson and T. Carrington* for studies in the kinetics of fast chemical reactions. That very high temperature can be obtained by

* T. Carrington, "Photoelectric Observation of the Rate of Dissociation of Dinitrogen Tetroxide behind a Shock Wave" Ph.D. Thesis, California Institute of Technology, 1952.

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shock tube is already demonstrated by Professor A. Kantrowitz* of the Cornell University.

A shock tube is simply a cylindrical vessel provided with diaphragms or valves at the ends. These valves are opened and closed at controlled instants. Consider the case of NO production again. The sequence of events in a shock tube can be depicted as that sketched in Fig. 5. (a) in that figure shows both ends of the tube open with the left side of tube connected to the low pressure exhaust vessel and the right side of tube connected to the fresh air supply. The fresh air is already heated to, say, 600°K by heat exchanger. The fresh air at low pressure, say, atmospheric, is flowing from right to left. The tube is now being charged. When the fresh air reaches the left end of the tube, the valve at that end closes. The momentum of air will then produce a compression wave which travels towards the right. When this compression wave reaches the right end, the right end valve closes. This compression process is shown in Fig. 5 (b). When the right end valve closes, or even slightly before that, the left end valve is opened again, but this time connection is made with a very high pressure source. Then a strong shock is send towards the right and compresses the fresh air charge in the tube to a very high pressure and temperature. This is shown as (c) in Fig. 5. When the shock wave reaches the closed right end valve, it is reflected and the pressure and temperature of the gas behind the reflected shock is still higher. This is shown as d) in Fig. 5. When the reflected shock moves to the left end of the tube, the left end of the tube is disconnected with the high pressure source and

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* E. L. Resler, S.-C. Lin and A. Kantrowitz, J. Appl. Physics, 23: 1390 (1952)

connected again with the low pressure exhaust vessel, and a strong expansion wave travels toward the right. This is shown as ⓒ in Fig 5. When the expansion wave reaches the right end of the tube, the right end valve is opened and connected with the fresh air supply. The cycle is then completed.

The average time intervals the gas is subjected to high pressure and high temperature can be changed by mainly varying the length of the tube. It is of the order of the length of the tube divided by the speed of waves. With a tube of the several feet in length, this interval at high temperature is of the order of milliseconds. Since the temperature is high, even this relatively short time interval should be sufficient to reach chemical equilibrium. In the particular case of charging with heated air, concentrations of nitric oxide considerably in excess of 1% should be attained. This concentration of NO is frozen by the expansion wave and is present in the exhaust gas. The processing the exhaust gas can proceed in the conventional manner.

It is seen from the above discussion that the shock tube method involves complicated valving and precise timing. This is undoubtedly a very difficult engineering problem. However, since the tube or any other parts of the system is not required to be continuously in contact with high temperature — high temperature only appears intermittently, the cooling problem is very greatly simplified. In addition, the method is very flexible in that heating time, the time interval between the shock wave and the expansion wave can be controlled. This immediately points to the possibility of freezing partially completed reactions, i.e., the possibility of catching reaction intermediates. Fundamentally then, this process of shock tube has higher potential than the gas turbine process.

The valve cycling can be accomplished by mounting several shock tubes

¶ Since the wave engine process has greater possibilities than the gas turbine process, research in gasdynamical production of chemicals should emphasize this aspect of the problem. 14

together and with the tubes parallel to a shaft, driven at a proper rotary speed. The valving is then done by appropriate port connections between the moving shock tubes and the stationary end compartments. This design brings the system very close to a class of machine called the wave engine by A. Kantrowitz,* which is currently under development by many concerns.** Therefore similar to the gas turbine process, the shock tube process may be called the wave engine process.

6. Research in the Gasdynamical Processes of Chemical Production

¶ In the previous discussion, the example of nitric oxide is chosen only for clarity in presentation of the concept of gasdynamical production of chemicals. As stated in the Memorandum by S. S. Penner (Appendix), more promising examples are, perhaps, the production of hydrazine N_2H_4 from ammonia NH_3 and the production of ethylene C_2H_4 or acetylene C_2H_2 from methane CH_4. But in any of these chemical reactions, the knowledge on their kinetics is lacking. Even if no attempt will be made to "understand" the chemical kinetics of these reactions, there is still the necessity of obtaining experimentally the data indispensable for estimating the feasibility, for defining the optimum conditions, and for designing the process of manufacturing any chemical. Therefore the first step in the research on the gasdynamical production of chemicals is to study promising reactions

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* A. Kantrowitz, "The Wave Engine" Paper presented before 1946 Annual Meeting of ASME, New York

** A similar device is the "Comprex" under development by the Brown Boveri Co. of Switzerland. See for instance, F. W. Barry, J. Appl. Mechanics (ASME), 17:47 (1950). The theoretical work in this paper is, however, incorrect.

by shock tube experiments. There is no need of cycling for a continuous process. The experiments can be performed with a simple tube with diaphragms and timing devices. The instrumentation should be such as to allow the analysis of the progress of chemical reaction, and the identification and the measurement of concentration of the species of molecules.

If the desired chemical compound is a product of thermodynamical equilibrium, then the upper limit of the yield can be theoretically computed as a function of the temperature and pressure at the end of compression in the shock tube. Thus a first estimate of the feasibility of gasdynamical production of these chemicals can be made. The production of nitric oxide is an example. If the desired chemical is a reaction intermediate, one must have first some experimental data. However as soon as preliminary experiments of the type described in the preceding paragraph are performed, theoretical analysis can answer the optimum conditions for yield. These theoretical work here should be tested and modified to finally give a specification for the design of pilot plant.

Appendix

SHOCK-TUBE EXPERIMENTS FOR THE
PRODUCTION OF CHEMICAL COMPOUNDS

S. S. Penner

(April, 1953)

During the last few years, shock-tube experiments have become a popular research tool for studies of gas properties at very high temperatures (A. Kantrowitz) and for quantitative measurements of very fast chemical reactions (N. Davidson and others).

A proposal for practical utilization of the high temperatures and very great rates of heating and cooling attainable in compression (and expansion waves) shocks has been made by H. S. Tsien. Briefly, it is proposed to investigate the chemical compounds which can be produced by passing compression (and expansion waves) shocks of controlled strengths through pure gases or (reactive) gas mixtures. It is apparent that the number of possible reaction products formed in pure gases or in gas mixtures is practically unlimited. Preliminary remarks concerning the production of important chemicals are summarized in the following paragraphs.

A. Hydrazine

Hydrazine offers attractive possibilities both as a fuel in bipropellant rocket engines and as a monopropellant or gas generant. The price of hydrazine in lots of 450 pounds has recently been reduced from $5.25 per pound to $3.15 per pound. * For large-scale applications in commercial peace-time transportation devices the cost would

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* Chemical and Engineering News, vol. 31, pp. 880-881, March 1953.

have to be reduced by at least a factor of ten before hydrazine could be considered to be competitive with gasoline. Several processes for large-scale manufacture of hydrazine are now under consideration. Some basic work on the production of hydrazine from ammonia, by using a glow discharge, has been described recently.[#] The fraction of NH_3 converted to N_2H_4 was found to be ~ 0.02 °/o under optimum conditions and corresponds to the production of 30 g of N_2H_4 per Kw. hr. at the cathode end of the discharge tube, a pressure of 5 mm of Hg, and a volume flow rate of 4.56 cc/sec at S.T.P. Although the results are encouraging, a great deal of additional work remains to be done before the discharge tube can be used commercially.

The work of Devins and Burton has established the fact that NH_3 can be converted, by using a suitable energy input, either to N_2H_4 and H_2 or to N_2 and H_2. The particular experimental conditions required in a shock tube for the optimum production of N_2H_4 (which is itself unstable with respect to NH_3, N_2, and H_2) must be determined experimentally. It is suggested to perform preliminary experiments with pure NH_3 and with mixtures of N_2 and NH_3 (1 mole of N_2 to 4 moles of NH_3 would correspond to the appropriate stoichiometric proportions for the production of N_2H_4) utilizing shocks of varying

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[#] J. C. Devins and M. Burton, paper No. 42 presented before the 123rd National Meeting of the American Chemical Society, Los Angeles, California, March 15 to 19, 1953.

strengths. The production of N_2H_4 is to be followed by performing quantitative absorption measurements at wavelengths corresponding to one of the normal vibration frequencies of N_2H_4.

B. Production of Hydrocarbons

The production of hydrocarbons in shock tubes from natural gas, pure methane, or any other readily available hydrocarbon, can be studied by utilizing methods similar to the procedure described for N_2H_4. It is clear that the principal experimental problem is one of identification and quantitative analysis of the reaction products. The experimental studies will require the use of an infrared- or mass-spectrograph. Provided the necessary instrumentation becomes available, the analytical work can be pursued by personnel now available at the Guggenheim Jet Propulsion Center.

C. Production of Special Chemicals

The complete exploitation of the shock tube as a research tool or commercial device for the manufacture of special chemicals utilizing a vast number of different reactive chemicals, appears to be a problem of considerable magnitude. In view of our almost total lack of knowledge concerning quantitative kinetics data for complex reactions, it is not possible to predict the nature of the reaction products which

will be producsd in shock tube experiments. On the other hand, it is to be expectedthat the experimental results will be of considerable importance fo: the development of modern chemical kinetics.

6.2

Calculations on a Jet - Pump

喷气射流泵

作者曾为美国军方写过一份带有密级的建议书，题为“Study of the Possibility of Using the Ejector Action of the Jet as a Source of Power for Driving the Propellent Pump”（采用射流的引射作用作为推进剂泵的动力源），书中建议应用射流引射的原理，利用火箭喷流引射空气，产生具有一定压力和流量的混合气体，作为驱动推进剂泵的动力源。这种动力系统包括从大气中获取空气的进口段，在混合段之前的一个火箭发动机以及混合段和扩压段等三个部分。

作者将上述用于推进剂泵的设想进一步具体化，形成一种新型的引射空气用的引射泵的方案，为设计这种引射泵进行了计算，写了题为“Calculations on a Jet - Pump”（喷气引射泵）的研究报告，原稿共有 13 页。何时写作和何处发表均不详。这里选印了手稿的前 6 页、一张引射泵的示意图以及有关泵的增压值的计算曲线图。

从喷气引射泵的示意图上可以看出，这种泵分为三个部分，即获取空气的锥形进口段、装有火箭发动机而使发动机喷出的废气与空气进行混合的混合段以及将混合气体减速而增压的扩压段。采用这种引射泵所能获得的增压主要取决于火箭发动机的推力。作者计算了不同推力下所获得的不同的增压值，主要的计算结果由第 2 页的表 1 给出，该结果也被绘制成曲线的形式。

Calculations on a Jet-Pump

Hsue-shen Tsien

Purpose: The purpose of this calculation is to investigate the possibility of using a rocket motor as the driving jet to pump air from atmospheric pressure to a pressure of 2 p.s.i. gage without exceeding a final temperature of 300°F, and to determine the approximate size of the rocket motor for an air flow of 150 c.f.s. free air.

Method: The calculation will be divided into three steps (Fig. 1)

(1) The Entrance Cone. In the entrance cone, the air is accelerated from rest to a velocity u_1 ft./sec, and the pressure decreased from p_0 to p_1.

(2) The Mixing Zone. In this zone, the air at u_1 ft/sec is mixed with the exhaust from the rocket motor in a pipe of constant cross-sectional area. At the end of mixing, the gas mixture is at a pressure p_2 and velocity u_2''.

(3) The Diffuser. In the diffuser, the gas mixture is slowed down from the velocity u_2'' to negligible velocity and the pressure is increased from p_2 to p_3. The diffuser efficiency is assumed to be 80%.

Therefore if T_3'' denotes the absolute temperature at the end of the diffuser, the conditions imposed are

$$T_3'' < (460+300)^\circ \text{ Rankine}$$

$$p_3 - p_0 = 2 \text{ p.s.i.}$$

Results. By assuming that u_1 = air velocity at the end of the contraction cone = 150 ft./sec., the results of calculation is given in the following table, and Fig. 2

Table I

Rocket Motor Thrust, Lbs.	Velocity at End of Mixing, ft/sec.	Final Air-Exhaust Temperature, °F.	Final Pressure p.s.i., Abs.	Gross Pressure Rise, p.s.i.
160.3	258.3	399.9	15.57	0.87
240.4	305.7	541.6	16.02	1.32
353.0	369.7	710.8	16.65	1.95

It is seen that the temperature of the air-exhaust mixture is generally too high, unless the gross pressure rise could be reduced.

Cont'd on p. 3.

2

Theory : The following subscripts are used:

$(\ )_0$ ~ condition of free air

$(\ )_1$ ~ conditions at the end of entrance cone and the beginning of mixing

$(\ )_2$ ~ conditions at the end of mixing and the beginning of diffuser

$(\ )_3$ ~ conditions at the end of diffuser.

Furthermore, the following notations will be used

p = pressure

u = velocity of air

u' = velocity of the exhaust from rocket motor

u'' = velocity of the air-exhaust mixture

T = absolute temperature of the air

T' = absolute temperature of the exhaust from rocket motor

T'' = absolute temperature of the air-exhaust mixture

ρ = density of the air

ρ' = density of the exhaust from rocket motor

ρ'' = density of the air-exhaust mixture

γ = ratio of specific heats for air

γ' = ratio of specific heats for the exhaust from rocket motor

γ'' = ratio of specific heats for the air-exhaust mixture

C_p = specific heat at constant pressure for air

C_p' = specific heat at constant pressure for the exhaust

C_p'' = specific heat at constant pressure for the air-exhaust mixture

A = flow cross-sectional area at the end of the entrance cone

A'' = flow cross-sectional area of the mixing zone

The increase in velocity of air in the entrance cone is quite small, hence the change in condition can be calculated by considering the air as an incompressible fluid. Then

$$p_1 = p_0 - \frac{1}{2}\rho_0 u_1^2 \qquad (1)$$

$$\rho_1 = \rho_0 \qquad (2)$$

$$T_1 = T_0 \qquad (3)$$

For the mixing zone, the equations to be satisfied are:

(1) The equation of continuity,

$$\rho_1 u_1 A + \rho_1' u_1' (A'' - A) = \rho_2'' u_2'' A'' \qquad (4)$$

(2) The equation of momentum

$$\rho_1 u_1^2 A + \rho_1' u_1'^2 (A'' - A) = \rho_2'' u_2''^2 A'' + A''(p_2 - p_1) \qquad (5)$$

(3) The energy equation

$$\rho_1 u_1 A \left\{ c_p T_0 \right\} + \rho_1' u_1' (A'' - A) \left\{ c_p' T_1' + \frac{1}{2} u_1'^2 \right\}$$

$$= \rho_2'' u_2'' A'' \left\{ c_p'' T_2'' + \frac{1}{2} u_2''^2 \right\} \qquad (6)$$

By using the equation of state for a perfect gas, Eq.(6) can be written as

$$\rho_1 u_1 A \left\{ \frac{\gamma}{\gamma - 1} \frac{p_0}{\rho_0} \right\} + \rho_1' u_1' (A'' - A) \left\{ \frac{\gamma'}{\gamma' - 1} \frac{p_1}{\rho_1'} + \frac{1}{2} u_1'^2 \right\}$$

$$= \rho_2'' u_2'' A'' \left\{ \frac{\gamma''}{\gamma'' - 1} \frac{p_2}{\rho_2''} + \frac{1}{2} u_2''^2 \right\} \qquad (7)$$

To simplify the equations, let

m = mass flow of air = $\rho_1 u_1 A$

m' = mass flow of the exhaust from rocket motor = $\rho_1' u_1' (A'' - A)$

m'' = mass flow of the air-exhaust mixture = $\rho_2'' u_2'' A''$

and μ = mass flow ratio = $\frac{m}{m'}$. Then

$$\mu = \frac{\rho_1 u_1}{\rho_1' u_1'} \frac{1}{\left(\frac{A''}{A} - 1\right)} \tag{8}$$

and

$$\gamma'' = \frac{\mu\gamma + \frac{c_p'}{c_p}\gamma'}{\mu + \frac{c_p'}{c_p}\frac{\gamma}{\gamma'}} \tag{9}$$

The three unknowns are ρ_2'', p_2 and u_2'' in Eqs. (4), (5) and (7). By eliminating ρ_2 and p_2, the equation for u_2'' is

$$\left(\frac{\gamma''}{\gamma''-1} - \frac{1}{2}\right)\left[\rho_1 u_1 + \rho_1' u_1'\left(\frac{A''}{A} - 1\right)\right] u_2''^2 - \frac{\gamma''}{\gamma''-1}\left[\rho_1 u_1^2 + \rho_1' u_1'^2\left(\frac{A''}{A} - 1\right) + \frac{A''}{A} p_1\right] u_2''$$

$$+ \rho_1 u_1 \cdot \frac{\gamma}{\gamma-1}\frac{p_0}{\rho_0} + \rho_1' u_1'\left(\frac{A''}{A} - 1\right)\left\{\frac{\gamma'}{\gamma'-1}\frac{p_1}{\rho_1'} + \frac{1}{2}u_1'^2\right\} = 0 \tag{10}$$

This is the quadratic equation for u_2''. There will be two roots of u_2'', one small and one large. In practice only the small root has significance, because the large root corresponds to an entropy decrease and is therefore in contradiction with the principles of thermodynamics. After having determined u_2'', p_2 and ρ_2'' can be calculated by the following expressions:

$$p_2 = p_1 + \frac{\rho_1 u_1}{\left(\frac{A''}{A}\right)}\left(u_1 - u_2''\right) + \rho_1' u_1'\left\{1 - \frac{1}{\left(\frac{A''}{A}\right)}\right\}\left(u_1' - u_2''\right) \tag{11}$$

and

$$\rho_2'' = \rho_1 \frac{u_1}{u_2''}\frac{1}{\left(\frac{A''}{A}\right)} + \rho_1' \frac{u_1'}{u_2''}\left\{1 - \frac{1}{\left(\frac{A''}{A}\right)}\right\} \tag{12}$$

6

The velocity of sound a_2'' in the gas-exhaust mixture is

$$a_2'' = \sqrt{\gamma'' \frac{p_2}{\rho_2''}} \qquad (13)$$

If η_d = diffuser efficiency, then

$$\frac{T_3''}{T_2''} = 1 + \frac{\gamma''-1}{2}\left(\frac{u_2''}{a_2''}\right)^2 \qquad (14)$$

and

$$\frac{p_3}{p_2} = \left[1 + \frac{\gamma''-1}{2}\left(\frac{u_2''}{a_2''}\right)^2\right]^{\frac{\gamma''\eta_d}{\gamma''-1}} \qquad (15)$$

The thrust of the rocket motor is $\rho_1' u_1'^2 (A'' - A)$ (16)

<u>Numerical Calculation</u>: For the numerical calculation, the following values will be used:

$c_p = 0.238$ B.t.u./lb. °F

$c_p' = 0.405$ B.t.u./lb. °F

$\gamma = 1.400$

$\gamma' = 1.265$

$u_1 = 650$ ft./sec.

$u_1' = 5500$ ft./sec.

$p_0 = 14.7$ p.s.i.

$\rho_0 = 0.002378$ slugs/ft.^{3}

$\rho_1' = 0.000530$ slugs/ft.^{3}

The parameter of the problem is then $\frac{A'}{A}$ or μ. The results of the calculation is shown in the Table I. and Fig. 2.

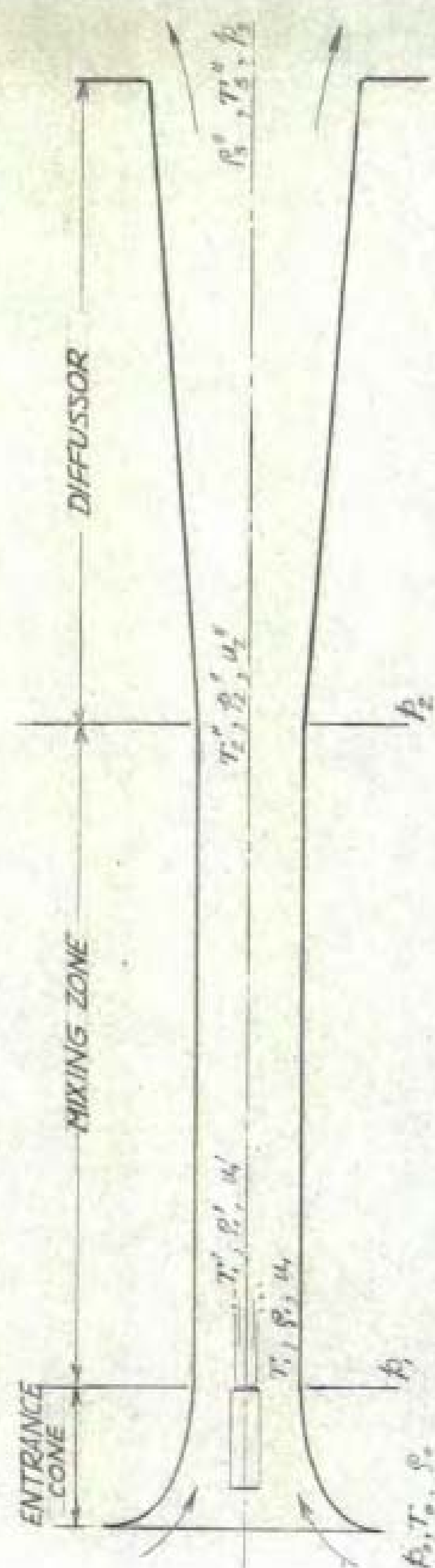
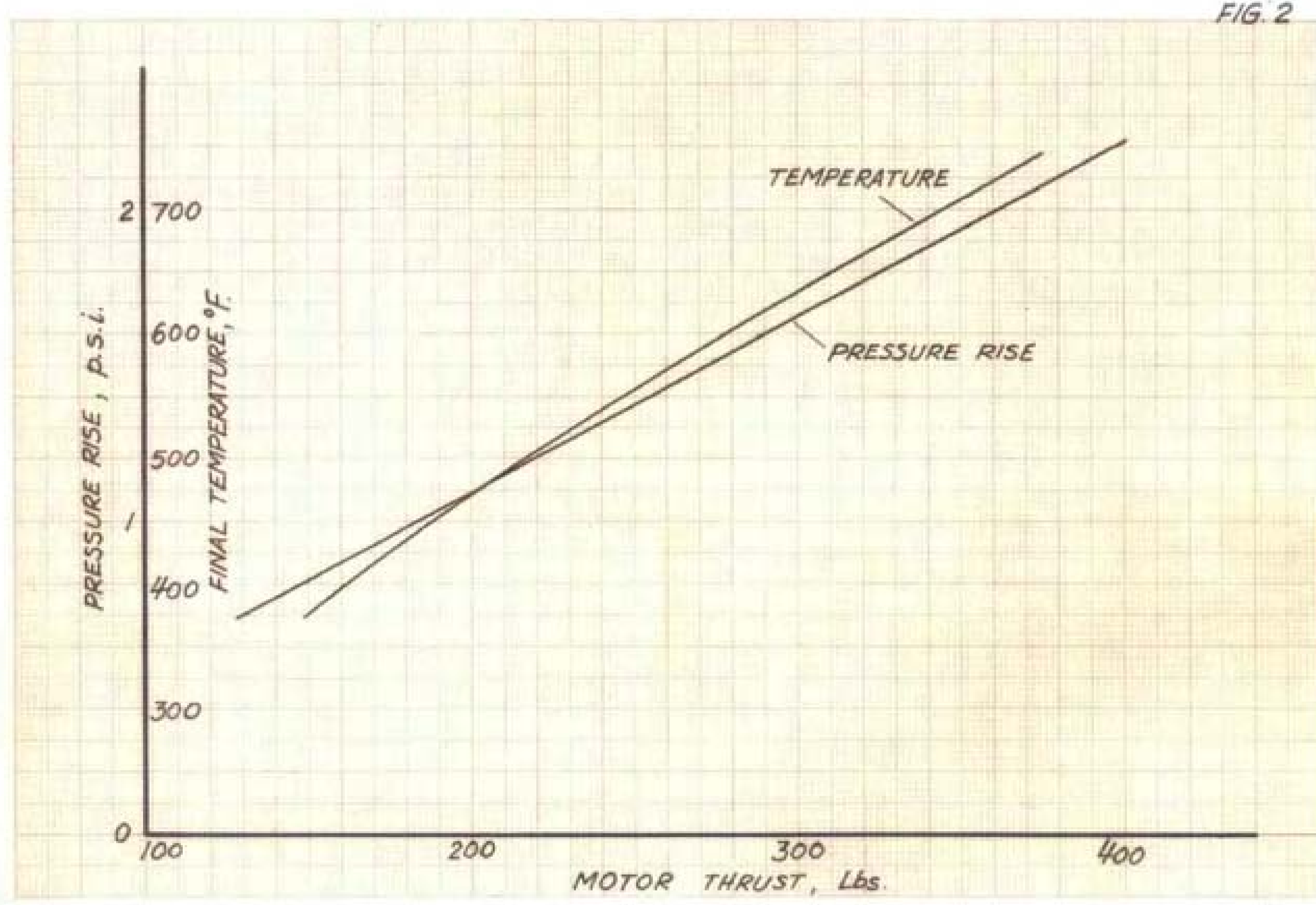


FIG. 1


FIG. 2
PRESSURE RISE, p.s.i.
FINAL TEMPERATURE, °F.
0
1
2
300
400
500
600
700
TEMPERATURE
PRESSURE RISE
100
200
300
400
MOTOR THRUST, Lbs.

6.3

Wind Mill for Power

产生动力的风车

作者在回国前曾经对风力发电作过分析研究，在他没有发表的手稿中有一份材料，其封皮上所写的题目是“Wind Mill for Power”，中文即是“产生动力的风车”，工作时间不详。

这份材料共有15页，是作者对风车特性所做的分析计算，针对不同的风速、不同的高度，计算可能达到的效率。在作者给出的实际算例中，所取的高度包括从海平面起到海拔8千米以上，可以想见，这里作者是从他祖国的自然条件出发，密切关注着祖国发展能源的迫切需要。

这里选印手稿中的10页，即手稿的前6页有关计算公式的推导、3页计算结果表格以及一张计算结果图。

作者在回国后继续进行风车发电的研究。他在1957年《科学记录》的创刊号上发表了“关于大型风力发电机”一文，提出了有关大型风车的新方案，即在风力大而风向改变不大的地方，建造一个风洞，把风车放在风洞里风速最大的地方，可以大大提高风车传动发电机的效率，这样的风车可称之为“风洞风车”。接着，他又派遣了一个研究组，深入新疆地区，进行了现场考察和实验。

在我国经济正在高速发展的今天，除了面临本来就很突出的能源短缺的问题，再加上严重的环境污染，越加说明作者所提倡的发展经济和干净的风力发电是多么的重要。

Windmill Calculation

1

Let D = drag

Q = torque

a = axial inflow factor

a' = rotational inflow factor

r = radius

R = radius of windmill

B = no. of blades

c = chord

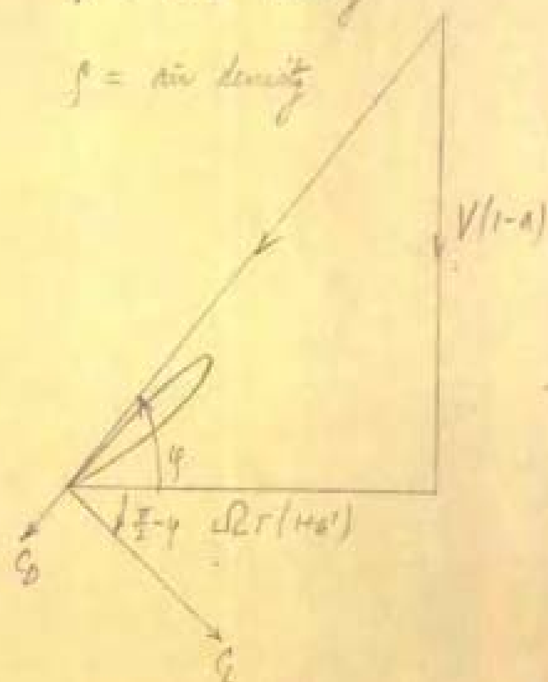
C_D = profile drag coefficient

C_L = lift coefficient

Ω = angular velocity

V = wind velocity

ρ = air density



From consideration of blade elements.

$$\frac{dD}{dr} = \frac{1}{2} B c \rho \Omega^2 r^2 (1+a')^2 (C_L \cos\varphi + C_D \sin\varphi) \sec^2\varphi \quad (1)$$

$$\frac{dQ}{dr} = \frac{1}{2} B c \rho \Omega^2 r^3 (1+a')^2 (C_L \sin\varphi - C_D \cos\varphi) \sec^2\varphi \quad (2)$$

From consideration of momentum:

$$\frac{dD}{dr} = 4\pi r \rho V^2 (1-a) a \quad (3)$$

$$\frac{dQ}{dr} = 4\pi r^3 \rho V \Omega (1-a) a' \quad (4)$$

where

$$\tan\varphi = \frac{V}{\Omega r} \frac{1-a}{1+a'} \quad (5)$$

To simplify the calculation, the following assumptions are made

(1) constant values of the interference factors a + a' along the blade;

(2) constant chord c along the blade;

(3) constant profile drag coefficient C_D along the blade;

(4) lift coefficient proportional to the sine of the angle of attack of the blade section, or

$$C_L = a_0 \sin(\varphi - \theta) \qquad (6)$$

where the blade angle θ is measured from the axis of zero lift of the blade section

(5) constant geometrical pitch H along the blade or

$$\tan\theta = \frac{H}{2\pi r} = \frac{hR}{r} \qquad (7)$$

The thrust and torque of the windmill will be expressed by appropriate non-dimensional coefficients, defined by the equations

$$D = D_c \pi R^2 \rho \Omega^2 R^2 \qquad (8)$$

$$Q = Q_c \pi R^2 \rho \Omega^2 R^3 \qquad (9)$$

These coefficients, for a given windmill, will be functions of the speed ratio

$$\lambda = \frac{V}{\Omega R} \qquad 10$$

and the corresponding efficiency of the windmill is $\eta = \frac{\Omega Q}{DV} = \frac{Q_c}{\lambda D_c}$ (11)

3

It is also convenient to write

$$\mu = \lambda \frac{1-a}{1+a'} \qquad (14)$$

and

$$\sigma = \frac{Bc}{\pi R} \qquad (13)$$

so that σ is the ratio of the blade area to the disc area & may be called the solidity of the windmill.

From (5) and (12), we have

$$r \tan\varphi = \frac{V}{\Omega R} R \frac{1-a}{1+a'} = \lambda \frac{1-a}{1+a'} R = \mu R \qquad (14)$$

$$\text{So} \qquad r = \mu R \cot\varphi \; , \qquad dr = -\mu R \csc^2\varphi \, d\varphi \qquad (15)$$

where μ is constant along the blade, and the integration of equations (1) & (2) can be made most conveniently by using φ as independent variable. Then

$$D_c' = \frac{D_c}{\sigma(1+a')^2} = \frac{\pi R}{Bc} \frac{1}{(1+a')^2} \frac{1}{\pi R^2 \rho \Omega^2 R^2}$$

$$\int_0^R \frac{1}{2} Bc \rho \Omega^2 r^2 (1+a')^2 \left[-a_0 \sin(\theta - \varphi) \cos\varphi + C_D \sin\varphi \right] \csc^2\varphi \, dr$$

$$D_c' = \mu^3 \int_{\varphi_1}^{\frac{\pi}{2}} \left\{ -\frac{1}{2} a_0 \frac{\sin(\theta-\varphi)\cos\varphi}{\sin^4\varphi} + \frac{1}{2} C_D \frac{1}{\sin^3\varphi} \right\} d\varphi \qquad (16)$$

4

$$Q_c' = \frac{Q_c}{\sigma(1+a')^2} = \mu^4 \int_{\varphi_1}^{\pi/2} \left\{ \frac{1}{2} a_0 \frac{\sin(\theta-\varphi)\cos\varphi}{\sin^4\varphi} - \frac{1}{2} C_0 \frac{\cos^2\varphi}{\sin^5\varphi} \right\} d\varphi \quad (17)$$

where the lower limit of integration φ_1 is defined by the equation

$$\tan \varphi_1 = \mu \quad (18)$$

and where

$$\tan \theta = \frac{h}{\mu} \tan \varphi \quad (19)$$

After integration

$$D_c' = \frac{1}{2} a_0 (\mu - h) J(h) + \frac{1}{2} C_0 K_1(\mu) \quad (20)$$

$$Q_c' = \frac{1}{2} a_0 \mu (\mu - h) J(h) - \frac{1}{2} C_0 K_2(\mu) \quad (21)$$

where

$$J(h) = \frac{1}{2} \left\{ \sqrt{1+h^2} - h^2 \log \frac{\sqrt{1+h^2}+1}{h} \right\} \quad (22)$$

$$K_1(\mu) = \frac{1}{2} \mu \left\{ \sqrt{1+\mu^2} + \mu^2 \log \frac{\sqrt{1+\mu^2}+1}{\mu} \right\} \quad (23)$$

$$K_2(\mu) = \frac{1}{8} \left\{ (2+\mu^2)\sqrt{1+\mu^2} - \mu^4 \log \frac{\sqrt{1+\mu^2}+1}{\mu} \right\} \quad (24)$$

Direct integration of the equations (3) & (4) gives

$$D = 2\pi R^2 \rho V^2 (1-a) a \quad (25)$$

$$Q = \pi R^4 \rho V \Omega (1-a) a' \quad (26)$$

and hence the axial & rotational interference factors are determined by the equations

$$\frac{a}{1-a} = \frac{\sigma D_c'}{2\mu^2}, \quad \frac{a'}{1+a'} = \frac{\sigma Q_c'}{\mu} \quad (27)$$

5

the calculation of the characteristics of any windmill, for which the values of σ and h are known, is now made by the following steps. Assuming suitable values of a_0 + C_D

$$a_0 = 5.50$$

$$C_D = 0.020$$

and starting with a suitable series of values of μ, the first step is to calculate the values of D_c' + Q_c' by means of (30) + (31). the interference factors a + a' are then determined by means of equations (35) + (36), + finally the characteristics of the windmill are derived from the equations

$$\lambda = \frac{1+a'}{1-a}\mu$$

$$D_c = \sigma(1+a')^2 D_c'$$

$$Q_c = \sigma(1+a')^2 Q_c'$$

It is interesting to investigate the location of highest Q_c $\quad Q_c \sim \sin(\varphi-\theta)$

$$\sin(\varphi-\theta) = \sin\varphi\cos\theta - \cos\varphi\sin\theta = \cos\varphi\cos\theta\left[\tan\varphi - \tan\theta\right]$$

$$= \cos\varphi\cos\theta\tan\varphi\left[1-\frac{h}{\mu}\right] = \frac{1}{\cos\varphi}\,\frac{1}{\sin\theta}\,\frac{V}{\Omega r}\,\frac{1-a}{1+a'}\left(1-\frac{h}{\mu}\right)$$

$$= \frac{1}{\sqrt{\tan^2\varphi+1}}\,\frac{1}{\sqrt{\tan^2\theta+1}}\,\frac{V}{\Omega r}\,\frac{1-a}{1+a'}\left(1-\frac{h}{\mu}\right)$$

6

So
$$\sin(\varphi-\theta) = \frac{\frac{V}{\Omega r}\,\frac{1-a}{1+a'}\left(1-\frac{h}{\mu}\right)}{\sqrt{\left\{\left(\frac{V}{\Omega r}\right)^2+1\right\}\left\{\frac{h^2}{\mu^2}\left(\frac{V}{\Omega r}\right)^2+1\right\}}}$$

$$\boxed{\sin(\varphi-\theta) = \frac{\frac{1-a}{1+a'}\left(1-\frac{h}{\mu}\right)}{\sqrt{\frac{h^2}{\mu^2}\left(\frac{V}{\Omega r}\right)^2+\left(1+\frac{h^2}{\mu^2}\right)+\frac{1}{\left(\frac{V}{\Omega r}\right)^2}}}}$$

Hence the maximum of $\sin(\varphi-\theta)$ occurs at

$$\frac{h^2}{\mu^2}\left(\frac{V}{\Omega r}\right)^2 = \frac{1}{\left(\frac{V}{\Omega r}\right)^2}$$

or
$$\boxed{\left(\frac{V}{\Omega r}\right)^2 = \frac{\mu}{h}}$$

therefore the maximum occurs

$$\boxed{\left[\sin(\varphi-\theta)\right]_{max.} = \frac{1-a}{1+a'}\,\frac{1-\frac{h}{\mu}}{1+\frac{h}{\mu}}}$$

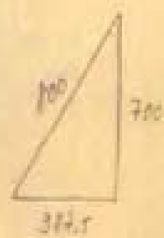
When $\left(\frac{V}{\Omega r}\right)^2 = \lambda^2 > \frac{\mu}{h}$

we have
$$\left[\sin(\varphi-\theta)\right]_{max} = \frac{\mu-h}{\sqrt{(\lambda^2+1)\left(\frac{h^2}{\mu^2}\lambda^2+1\right)}}$$

7

Extension of the Table given by H. Glauert, RM 1342, (1930)

①	②	③	④	⑤	⑥	⑦	⑧
μ	μ^2	$\sqrt{1+\mu^2}$	$\frac{③+1}{①}$	log ④	$J(\mu)$	$K_1(\mu)$	$K_2(\mu)$
1.0	1.000	1.414	2.414	0.881	0.2665	1.1425	0.4201
1.1	1.210	1.486	2.260	0.815	0.2500	1.3597	0.4471
1.2	1.440	1.562	2.132	0.757	0.2360	1.5913	0.4755
1.3	1.690	1.640	2.030	0.708	0.2218	1.8437	0.5077
1.4	1.960	1.721	1.944	0.665	0.2088	2.1121	0.5326
1.5	2.250	1.803	1.868	0.624	0.1995	2.4053	0.5630
1.6	2.560	1.887	1.803	0.589	0.1896	2.7158	0.5921
1.7	2.890	1.972	1.748	0.5585	0.1790	3.0482	0.6223
1.8	3.240	2.060	1.700	0.5305	0.1706	3.4010	0.6532
1.9	3.610	2.146	1.656	0.5044	0.1632	3.7671	0.6817
2.0	4.000	2.234	1.616	0.4800	0.1570	4.1560	0.7155



$$\lambda = \frac{700}{387.5} = 1.812$$

$$R = 1.5', \qquad \Omega = \frac{387.5}{1.5} = 258 \text{ rad/sec.}$$

$$= 2,466 \text{ r.p.m.}$$

$$Q = \frac{4.5 \times 550}{258} = 9.7 \text{ ft-lbs}$$

$$9.7 = Q_c \, \pi (1.5)^5 \times (258)^2 \times 0.002378 = Q_c \, \pi \times 7.6 \times 6.65 \times 2.378$$

$$Q_c = \frac{9.7 \times 10^{-3}}{\pi \times 0.76 \times 0.665 \times 0.2378} = 0.1001$$

$c = 0.5'$, $B = 3$, $\sigma = 0.212$

1

$\ell = 1.3, \quad J(\ell) = 0.2318, \quad \sigma = 0.22$

	μ			
	1.5	1.6	1.7	1.8
$\frac{1}{2} a_0 (\mu - h) J(\ell)$	0.121990	0.182985	0.243980	0.304975
$\frac{1}{2} C_0 K_1(\mu)$	0.024253	0.027158	0.030462	0.034010
D_c'	0.1460	0.2101	0.2745	0.3390
$\frac{1}{2} a_0 \mu (\mu - h) J(\ell)$	0.182985	0.292776	0.41477	0.54896
$-\frac{1}{2} C_0 K_2(\mu)$	−0.00563	−0.00593	−0.00622	−0.00653
Q_c'	0.1774	0.2868	0.4086	0.5424
$\sigma D_c' / 2\mu^2$	0.007138	0.009028	0.010448	0.011509
a	0.007067	0.008967	0.010340	0.011278
$\sigma Q_c' / \mu$	0.03602	0.03944	0.05088	0.06629
a'	0.03672	0.04106	0.05583	0.07100
λ	1.5511	1.6807	1.8137	1.9500
Q_c	0.0411	0.0684	0.1002	0.1369
η %	78.2	81.2	82.1	82.0
λ^2	2.4059	2.8248	3.2895	3.8025
$[\sin(\varphi - \theta)]_{MAX}$	0.0647	0.0907	0.1116	0.1230

$C_L = 1.615$

for $\lambda = 1.812, \quad Q_c = 0.1001, \quad \eta = 82.1\%$

Altitude = 40,000 ft.

$D = 41.6$ lb.

12

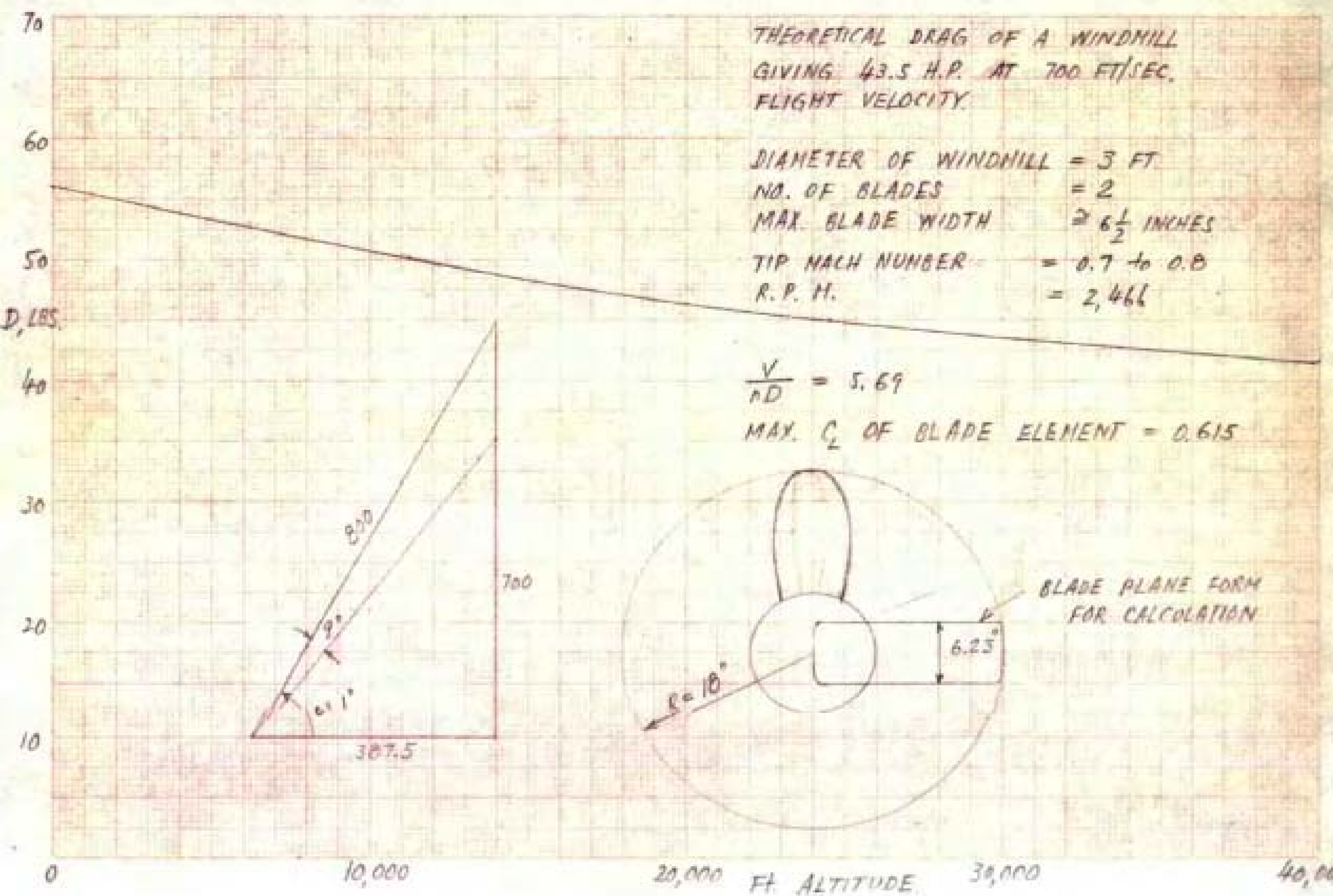
$h = 1.7$, $J(h) = 0.1790$, $\sigma = 0.22$

	μ			
	1.7	1.8	1.9	2.0
$\frac{1}{2}a_0(\mu-h)J(h)$		0.049225	0.098450	
$\frac{1}{2}C_0K_1(\mu)$		0.034010	0.037671	
D_c'		0.0832	0.1361	
$\frac{1}{2}a_0\mu(\mu-h)J(h)$		0.088605	0.187055	
$-\frac{1}{2}C_0K_2(\mu)$		-0.00653	-0.00684	
Q_c'		0.082075	0.18022	
$\sigma D_c'/2\mu^2$		0.002825	0.004147	
a		0.002817	0.004130	
$\sigma Q_c'/\mu$		0.01031	0.02087	
a'		0.01042	0.02131	
λ		1.8039	1.9425	
Q_c		0.01843	0.0414	
η		54.1	67.9	

$\lambda = 1.812$, for sea level $Q_c = 0.0245$, $\eta = 61.1\%$

$D = $ 56 lbs.

$\lambda =$


THEORETICAL DRAG OF A WINDMILL
GIVING 43.5 H.P. AT 700 FT/SEC.
FLIGHT VELOCITY
DIAMETER OF WINDMILL = 3 FT.
NO. OF BLADES = 2
MAX. BLADE WIDTH ≅ 6½ INCHES
TIP MACH NUMBER = 0.7 to 0.8
R.P.M. = 2,466
V/nD = 5.69
MAX. C_L OF BLADE ELEMENT = 0.615
BLADE PLANE FORM FOR CALCULATION
6.25"
ε=10°
800
700
9°
61.1°
387.5
D, LBS.
70
60
50
40
30
20
10
0
10,000
20,000
30,000
40,000
FT. ALTITUDE

6.4
Thermonuclear Power Plants
热核电站

在50年代中期，人们对核电站的兴趣集中在裂变反应器上，只有少量的文章讨论聚变反应器，但是作者当时已经敏锐地意识到，世界上裂变燃料的矿产资源极为有限，而聚变燃料相对地说却极大地丰富，研究和发展聚变能源有着光明的前景，应该把热核电站的研究提到日程上来。于是，作者从工程科学的角度探讨了热核电站的特性以及技术设计中的几个基本问题，诸如：热核反应速率，反应器燃烧室的冷却散热问题，燃烧室和气体透平、分离器、热交换器以及气体压缩机等组成的循环系统等等。作者在回国前不久完成了题为“Thermonuclear Power Plants”（热核电站）一文，并委托Frank E. Marble（F·马勃）将论文投送《Jet Propulsion》（喷气推进学报）。在作者回国后的第二年，即1956年，论文发表了。

这里选印上述论文手稿的前3页和后3页。

Thermonuclear Power Plants

(I)
Introduction

1.1 Preliminary Discussion

The thermonuclear reaction we have in mind is



$$_1H^2 + {}_1H^2 \to {}_2He^3 + {}_0n^1 \qquad (1.1)$$

This reaction is singled out because of the availability of deuterium and the relatively high reaction rate. Since the mass of $_0n^1$ is 1.008982, of $_1H^2$ is 2.014735, and of $_2He^3$ is 3.016977 (see cf. Kaplan, p. 232), each single event specified by (1.1) generates

$$2\times 2.014735 - (1.008982 + 3.016977) = 0.003511 \text{ amu} \qquad (1.2)$$
$$= 3.27 \text{ Mev.} = 5.24\times 10^{-6} \text{ ergs.} = 5.24\times 10^{-13} \text{ watt-sec.}$$

energy. Therefore $2\times 2.014735 = 4.029470$ gr. of deuterium gas will produce when completely "burned" into $_2He^3$ and neutron

$$5.24\times 10^{-13}\times 6.02472\times 10^{23} \text{ watt-sec.}$$
$$= 3.15\times 10^{11} \text{ watt-sec.} = 87700 \text{ kw-hr.}$$

Hence the energy production by the thermonuclear fusion of deuterium is

$$21,800 \text{ kw-hr./gr.} = \underline{9,870,000 \text{ kw-hr./lb.}} \qquad (1.3)$$

The present rate of electric energy production in the United States is approximately 500×10^9 kw-hr. per year. Thus assuming a thermal efficiency of 25%, this annual energy can be supplied by burning approximately

$$\frac{1}{0.25}\times 5\times 10^{11}\times 10^{-7} = 2\times 10^5 \text{ lb.} = 100 \text{ tons of } D_2 .$$

Now the relative abundance of deuterium and ordinary hydrogen is

roughly 1:7,000. Thus 100 tons of D_2 means 700,000 tons of hydrogen or 6,300,000 tons of water. The same amount of energy if produced by coal (at approximately 1 lb. per kw-hr.) requires 250,000,000 tons of coal. Therefore roughly, <u>1 unit weight of deuterium is equivalent to 2.5×10^6 units of weight of coal</u>, or <u>1 unit weight of water is equivalent to 40 units of weight of coal</u>. The total amount of water on earth is approximately 1.4×10^{24} gr. $= 1.54 \times 10^{18}$ tons. This is equivalent to 6×10^{19} tons of coal, an amount far exceed fossil fuel reserve, and uranium and thorium reserve combined.

1.2 <u>Separation of Deuterium from Water</u>

Three methods of isotope separation have been applied to obtain deuterium from water: 1) the electrolytic method, 2) distillation and 3) the chemical exchange method. Electrolysis has been applied with great success to the separation of deuterium. It was found experimentally that when an aqueous solution is electrolyzed the lighter isotope of hydrogen is evolved more rapidly than the heavier isotope, and the residue in the electrolysis cell is enriched in deuterium. In the industrial manufacture of oxygen and hydrogen, cells containing potassium hydroxide solutions are operated for long periods of time without changing the electrolyte in the cells; water is added periodically to make up for the water electrolyzed. The residue from these cells can be used as the feed material for further purification, and the method consists in electrolyzing a large volume of such water down to a small residue. A dilute solution of sodium hydroxide, from industrial cells, is electrolyzed with a large current and nickel electrodes until all but about 1/10 of the water has been decomposed into hydrogen and oxygen. The residue is very concentrated

in alkali and is partially neutralized by the addition of CO_2. The enriched water is then distilled off and the distillate goes to the next stage of electrolysis, where the procedure is repeated. About five to seven stages are needed to yield water highly enriched in deuterium. In the later stages, the hydrogen gas which is evolved is rich in deuterium, and it is therefore burned to form water and returned to the electrolytic cells. To obtain water with 99% deuterium, it is usually necessary to electrolyze ordinary water until it is reduced to 10^{-5} of its original volume. An indication of the rate of concentration can be obtained from Table 1.1, (Taylor, Eyring and Frost, "Technique for the Electrolytic Production of D_2O, J. Chem. Phys. 1: 823, 1933); the initial electrolyte

Table 1.1

The Concentration of Deuterium by Electrolysis

Stage	Volume of electrolyte	Density gr./cm³.	% of D
I	2300 liters	0.998	0.03
II	340 "	0.999	0.5
III	52 "	1.001	2.5
IV	10 "	1.007	8
V	2 "	1.031	30
VI	420 c.c.	1.098	93
VII	82 c.c.	1.104	99

was from a commercial cell and the deuterium abundance was about 0.03%.

The method of distillation for separating deuterium is based upon the fact that the normal boiling point of D_2O is 1.4°C higher than

chamber, and the remaining large fraction is transmitted directly to the wall. The crucial energy production within the chamber is that part kept within the chamber. To prevent the reaction to be quenched, the heat "kept" must be equal to the heat conducted and radiated to the wall by the conventional processes.

3.2 Effective Energy Production in the Chamber

Let E_1 be the energy of $_2He^3$ and E_2 be the energy of $_0n^1$ in (1.1). Then if E is the total energy,

$$E_1 + E_2 = E \qquad (3.1)$$

But balance of momentum requires

$$3E_1 = E_2 \qquad (3.2)$$

Or

$$E_1 = \frac{1}{4}E = 0.818 \text{ Mev}$$

At 100 atm and 10^8 K, the effective energy production due to (1.1) is thus 0.5 cal./cm^3 sec.

Let E_3 be the energy of $_2He^4$ and E_4 be the energy of γ in (1.9). The sum of E_3+E_4 is

$$E_3 + E_4 = 20.57$$

$$\sqrt{8E_3} = \frac{E_4}{c}$$

Thus approximately

$$E_4 \cong 20.57 \text{ Mev.}$$

and

$$E_3 = \frac{1}{8}\frac{E_4^2}{c^2} = E_4 \times \frac{20.57}{8 \times 931.1} = 0.0568 \text{ Mev.}$$

Hence the influence of recoil energy of He^4 ions are completely unimportant.

Let us take the mass absorption coefficient of 20 Mev. γ-ray as 0.03 cm²/gr. At 10^{8} °K and 100 atm, the density of ionized deuterium is

$$\frac{1}{22400} \times 100 \times \frac{273}{10^{8}} \sim 10^{-8} \text{ gr./cm}^{3}.$$

If the path length is 25 m = 2500 cm. the fraction of γ-ray energy absorbed is

$$0.03 \times 2500 \times 10^{-8} \cong 10^{-6}$$

Even if the energy absorbed is multiplied by a factor 10^{2} due to cooler gas near the wall, the effective energy production will not be greatly different from 0.5 cal/cm³ sec.

Let the relatively cool deuterium near the wall has a heat conduction coefficient equal 0.0004 cal/cm. sec. °K. The temperature gradient, assuming a central zone of 30 m in 50 m diameter chamber, is

$$10^{8} / 1000 = 10^{5} \text{ °K./cm.}$$

The cooling rate is thus

$$q = 40 \text{ cal/cm}^{2} \text{ sec.}$$

The effective energy production rate is

$$0.5 \times 1500 = 750 \text{ cal/cm}^{2} \text{ sec.}$$

This corresponds to a black body radiation at 6000°K

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6.5
第二次世界大战末期对德国航空和火箭研究的调研报告（1945年5月）

1944年，即第二次世界大战结束前一年，美国陆军航空兵（The Army Air Force）的 Henry Arnold（亨利·阿诺德）将军已把目光投向战后的一个很长时期。是年9月，他单独与 Theodore von Kármán（冯·卡门）会晤，要 von Kármán 组织一个科学咨询团（Science Advisory Group），为今后20－50年美国空军的长远发展提供科学研究工作的蓝图。von Kármán 很快组建了这个咨询团，并请钱学森参加这个团的核心组。1945年春，钱学森参观了美国几个有名的实验室，如 RCA 实验室、NACA（美国国家航空委员会）、JPL（喷气推进实验室）等，评估美国航空研究和发展的水平和趋势。1945年3月，欧洲战场上的德军全面崩溃，Arnold 又向 von Kármán 建议，到德国去看看他们究竟在航空和火箭的研究和发展方面走得有多远，去查问德国科学家并视察他们的实验室，搜集第一手资料，顺便考察英、法、瑞士、瑞典等欧洲国家的研究情况。4月底，钱学森等随同 von Kármán 飞往欧洲。在德国，钱学森查问了德国火箭研究的最高权威 von Braun（冯·布劳恩）和研究 V－2 火箭的著名理论家 Rudolf Hermann（鲁道夫·赫尔曼）等人，视察了美军发现的德国人的秘密实验室和 V－2 火箭工厂，查阅了德国人有关火箭和空气动力学的秘密研究报告。5月间，钱学森写出了一系列调研报告，反映德国人在飞机、火箭、炸弹等多方面的发展状况。在现存的钱学森手稿中便有他在1945年5月17－21日所写的部分报告的底稿，计有题为 “Arrow－Wing（Pfeilflügel）”（箭形机翼），“Rockets”（火箭），“Gasdynamics with supersonic velocities”（超声速气体动力学），“Ram-jet”（冲压发动机），“Aeropulse”（脉冲式空气喷气发动机），“Liquid explosive bombs”（液体炸药炸弹），“Installation of turbojets in an airplane”（飞机上的喷气涡轮发动机的安装问题）等7篇。这些报告乃是对

德国调查研究的结果。6 月 20 日，钱学森结束欧洲之旅，回到华盛顿。

1945 年，以 von Kármán 为首的科学咨询团，为美国陆军航空兵（The Army Air Force）写了题为 " Toward New Horizon "（迈向新高度）共 9 卷的带有展望和规划性的报告，为发展美国现代化空军提供远景发展蓝图。在这一系列报告的第 3 卷，钱学森以上述调研报告为基础，介绍了战时德国和瑞士在航空研究领域中的发展情况，取名为 " Reports on the Recent Developments of Several Selected Fields in Germany and Switzerland"（关于德国和瑞士在某几个领域近期发展情况的报告）。

下面将分别选印作者在 1945 年 5 月所写 7 篇报告的部分手稿。

6.5.1

Arrow – Wing

箭形机翼

作者在 1945 年 5 月 17 日所写的题为“ Arrow – Wing ”(箭形机翼）的调研报告手稿共 6 页。手稿说明了箭形机翼的基本原理、战时德国研究箭形机翼最活跃的单位和专家以及在空气动力学方面的实验研究结果，并且指出了箭形机翼所存在的缺点。这里仅选印原稿的首页。

当时一般机翼的翼展方向是和机身垂直的。当飞机速度不断增加时，空气的可压缩效应越来越显著，这种效应可以用马赫数（即飞行速度与空气声速的比值）来表征。当马赫数达到某一数值(一般在 0.74 左右)时，机翼的空气动力学特性发生根本的变化，升力骤减而阻力骤增，这时的马赫数称为临界马赫数。为了避免机翼在高速飞行时的效率发生大的损失，若把机翼的形状作一改变，即把翼展的方向从垂直于机身的方向朝后方折转一个角度，而作成箭形机翼（也称后掠式机翼），可以把临界马赫数提得更高。

作者在 1944 年 4 月随 Theodore von Kármán（冯・卡门）率领的科学咨询团飞往德国之前，曾经和他的同事们讨论过不久之前美国国家航空委员会的 Robert Jones 对箭形机翼所做的理论分析，认为 Jones 的理论有道理，但是需要取得实验结果的支持。然而当作者到达德国进行了实地考察以后，发现德国人在大战期间对箭形机翼的研究远比美国人深入。1940 年，H. Ludwieg 所写的报告给出了有关翼展有限的箭形机翼的空气动力学特性的风洞实验结果，清楚地说明了箭形机翼在高速飞行中可以使阻力大为减小；1942 年，G. Koch 所写的实验研究报告说明，粘性效应是可以忽略的，可以用理想流体的理论来估算导致失速现象的临界马赫数，等等。

在报告的最后一段，作者谈到了箭形机翼所存在的问题。由于机翼表面上的压力分布与通常直形机翼不同，气流绕过机翼时较早发生分离现象，导致飞行稳定性的降低，这一问题需要今后进一步的研究，以便做出改进。

1

Arrow-wing (Pfeilflügel)

(H. S. Tsien)

May 17, 1945

I. Introduction. As the flight velocity of the aircraft is increased, the effects of the compressibility of the air become more and more pronounced. It is well-known that these effects can be measured by the single parameter called the Mach number. The Mach number is the ratio of the flight velocity to the sound velocity. If the Mach number approaches one, the aerodynamic characteristics of a wing are radically changed by a decrease in lift and an increase in drag. For the conventional wings used in the present-day aircraft, the radical change occurs generally at a Mach number of 0.74. To avoid the loss of aerodynamic efficiency of the wing at higher speeds, this critical Mach number must be pushed to higher limits by new designs.

The purpose of the arrow wing (Pfeilflügel)(3)(1) is to raise the critical Mach number. It was first suggested for supersonic flight with Mach number greater than unity by A. Busemann (Ref. 1) but the idea was adopted for subsonic flight velocity by A. Betz. (Ref. 2) In general the critical number with the same airfoil section can be raised by this means to a Mach number 0.1 higher. For instance if the straight wing has a critical Mach number of 0.74, then the arrow wing has a critical Mach number of 0.84. Therefore this idea constitutes a most important advance in applied aerodynamics and should be intensively exploited.

→ Insert.

II. The Fundamental Principle

For the moment, consider the air as non-viscous and therefore that there is no boundary layer exists over the surface of the wing. Fig. 2 shows a wing of infinite span placed in an air stream of Mach number 0.6 with the span of wing perpendicular to the flow direction. Since the Mach number is below the critical value, the aerodynamic efficiency of the wing is high

6.5.2

Rockets

火箭

在作者所参加的科学咨询团随着美军进入德国向前推进的过程中，连续不断地发掘和抢救了德军撤退时隐藏或企图销毁的大量研究报告和实验装备。他们发现，德国人为了准备战争，大概从1936年开始，便大规模地展开火箭的研制工作。德国人具有非常明确的军事目的，主要是为了给在高空飞行的歼击机突然加速、为缩短飞机起飞或降落的跑道、为发射鱼雷等用途而将火箭用作动力推进装置等。

作者在1945年5月18日所写的题为“Rockets”（火箭）的调研报告手稿共有15页。在这份报告中，作者介绍了德国人所研制的固体推进剂火箭、固体-液体推进剂火箭和液体推进剂火箭等三类火箭，包括推进剂的组分和性能以及火箭的结构和性能等。这里仅选印手稿的首页。

固体推进剂已经用在德国火箭炮（artillery rockets）上，它是一种用硝化棉和二硝酸二甲醇为主要原料的混合物，被压制成多孔条状的推进剂。为了制造更轻的火箭，在降低燃烧室压力和增加燃烧时间方面作了努力，研制了一种随燃烧室压力大小而开闭的调节器，可以使周期间隙性的燃烧转变为平稳连续性的燃烧。

所谓固体-液体推进剂火箭有两种类型，一种是先把固体碳压在燃烧室里，用时将氧化剂 N_2O 注进去；另一种则是把氧化剂硝基过氯酸盐晶体（nitrogyl perchlorate $NOClO_4 \cdot H_2O$ 晶体）和碳混合后压在燃烧室里，然后将液态燃料 NH_3 注进去。这类火箭的小型实验是成功的。

德国人在研制液体火箭方面，开始是用液氧加酒精作为推进剂，而后来则着重采用过氧化氢类型和硝酸苯氨类型的推进剂。在 A. Busemann 指导下由 E. Sänger 在 Müden 附近的 Fassberg 建造了一个大规模的试验装置，计划目标是研制成200 000磅的火箭，采用液氧加燃油作为推进剂。到了

1944年，该工作改由Grumbt负责。一个重要进展是提出了一个冷却燃烧室的方案，实验说明燃烧室采用多孔材料，将冷却液体渗入燃烧室壁面而形成一层液膜，可起到有效的冷却作用，作者认为这一方案应当大力加以研究。

1

Rockets

(H. S. Tsien)

May 11, 1945

I. Introduction. The intensive development of rocket in Germany started approximately in 1936 when the preparation for war was pushed in earnest. The main applications planned were:

a) Main propulsion power plant for pursuit and fighter airplanes with high rate of climb and high horizontal speed at very high altitudes

b) Auxiliary power plant for assisted take-off to shorten ground run, for increasing the rate of climb, and for braking during landing on small fields

c) Propulsion for munitions such as anti-aircraft rockets, glide bombs, accelerated bombs and projectiles

d) Propulsion for torpedoes and braking the torpedoes launched by fast flying aircrafts before entering water

e) Gas generation by rocket propellants for rotating or translational drive.

To develop the rockets for such purposes, the German industries and research institutions were mobilized. The most active ones were the following:

1) Luftfahrtforschungsanstalt Hermann Göring, Braunschweig.
Small research installation at Volkenrode (Noggerath and Schmidt) and large installation at Fassberg. (A. Busemann director, Gramatt in charge, also Winkler)

2) Luftfahrtforschungsanstalt München
Planned extensive installation at Ottobrunn. (O. Lutz, director)

3) Heeresanstalt, Peenemünde. Later moved to Nordhausen

4) Rheinmetall-Borsig, A.G., Berlin-Marienfelde (Solid propellants)

5) Fa. Wilhelm Schmidding, A.G., Bodenbach (Liquid propellants)

6) H. Walter, K.G., Kiel (Hydrogen peroxide propellant)

6.5.3

Gasdynamics with Supersonic Velocities

超声速气体动力学

作者在1945年5月20日所写的题为"Gasdynamics with Supersonic Velocities"（超声速气体动力学）的调研报告手稿共有5页。战时德国在超声速流动方面的研究，主要集中在以下三个方面：a）壳体和导弹的空气动力学特性；b）与脉冲式发动机和冲压式发动机的设计有关的流动问题；以及c）爆裂波或击波。手稿主要介绍德国人在壳体和导弹的实验和理论研究以及超声速风洞设计方面的情况。这里仅选印手稿的首页。

德国人的实验规模很大，主要做打靶试验和风洞试验。在Volkenrode的Hermann Göring航空研究实验室的打靶试验风洞，长度为400米，发射端的直径为5.4米，靶端的直径为7.6米，抽真空所达到的压力为0.05大气压，相当于24公里高空的状态。风洞试验的主要目的是，考查以风洞作为测量壳体的空气动力学特性的手段是否可靠。由AVA和HAP两大实验室所做的风洞试验结果并不一致，说明风洞试验的主要困难在于，反射弓形波的存在和支架的存在引起旋涡的畸变的影响以及不同的雷诺数对壳体表面摩擦的影响。

德国人在理论研究方面，用特征线法和线性化近似满意地解决了尖头旋转体和导弹的超声速流动问题，用特征线法计算了无粘流体的绕流问题，在离开圆滑表面的附近发现有击波形成，作者认为这一研究对说明击波和边界层的相互作用极有帮助。

在超声速风洞的设计方面，德国人倾向于采用方形试验段，一方面易于避免击波的形成，另一方面也有利于模型的支撑和天平的放置。

Gasdynamics with Supersonic Velocities

(H.S. Tsien)

May 20, 1945

I. Introduction. The main German effort of investigating the supersonic flows seems to be concentrated on the following subjects:

a) Aerodynamic characteristics of shells and missiles

b) Flow problems in connection with the design of aeropulses and ramjets at high speeds

c) Detonation or shock waves

The item b) will be treated in the report on the aeropulses and the ramjets. The item c) is essentially a combined aerodynamic and chemical phenomena. The aerodynamic aspect of the problem is reduced to the problem of cylindrical and spherical shock waves. Only theoretical work was done on this subject.

II Experimental Investigation of Shells and Missiles

The experimental investigation of shells and missiles was carried out by both firing and wind tunnel testing. For the firing tests, the most interesting equipments are the two firing tunnels at the Luftfahrtforschungsanstalt Hermann Göring at Volkenrode. One tunnel is 400 m long, with 5.4 m diameter at the firing end and 7.6 m diameter at the target end. It can be evacuated to 0.05 atmosphere corresponding to 72,200 ft altitude. The evacuation is done by a 500 kW exhauster, and complete evacuation is done in 4 hrs. Thus ballistic measurements can be made at extremely reduced air density. In addition there is a cross wind firing channel which is 30 m long and 0.6 m wide. Here a cross wind of up to 200 m/sec is created by discharging from free air to an evacuated tank of 5000 m^3 volume. At this cross wind velocity, the test duration is 0.6 second, sufficient for ballistic tests.

6.5.4

Ramjet

冲压式发动机

作者在 1945 年 5 月 20 日所写的题为 “Ramjet” (冲压式发动机) 的调研报告手稿共有 3 页。手稿叙述了战时德国研制冲压式发动机的情况，包括为改善燃烧性能、增加推力、降低外部阻力所做的模型试验以及飞行试验两个方面。这里仅选印手稿的首页。

冲压式发动机的模型试验是在 Hermann Göring 航空研究实验室进的。随着燃料注入量的增加，净推力先是不断增大，随后则变小；而最大净推力则随飞行速度而单调增加。随飞行马赫数的增加，净推力与动压和迎风面积的乘积的比值减小，燃烧也变得非常不平稳。在超声速风洞中所做的扩压器试验说明，在发动机的正前方形成一个正击波。为了减小压力损失，在进口部分引入一个中心锥，并使之突出在发动机的前方，头部不再出现正击波而只有一个斜击波，从而提高了扩压器的效率。

德国人曾经在轰炸机的顶部安装了直径为 2 米的冲压式发动机，在实地飞行中发现燃烧不平稳；此外，飞行员报告说，在高速飞行中飞机所受到的推力有明显的增加。

为了改进燃烧的平稳性和提高燃烧效率，德国人对燃烧室的内部结构形式作了多种尝试，例如在燃烧室内采用分布式的多头燃烧器，让可燃物流过挡板后再行燃烧，又如将燃烧室的内部设计成具有逐级扩增的台阶，目的都在于维持火焰的稳定燃烧，等等。

Ramjet

(H.S. Tsien)

May 20, 1945

1. Fa. Walter Company of Kiel has designed a ramjet model which was tested at the Luftfahrtforschungsanstalt Hermann Göring (LFA), Volkenrode, (the data are given in Untersuchungen und Mitteilungen Nr. 3014, "Untersuchungen an L-Triebwerk der Fa. Walter-Kiel in Hochgeschwindigkeitskanal A9 a der LFA", May 1, 1943). The cold unit has a drag coefficient equal to 0.3. The net thrust increases first with increase in fuel injection, but then it decreases again. The maximum value of net thrust increases with velocity. However the net thrust coefficient (net thrust divided by the product of dynamic pressure and the frontal area) decreases from 0.4 to 0.3 from Mach number 0.42 to 0.75. The unit can be ignited with wind tunnel velocities up to 700 ft/sec. The burning can be controlled by regulating the fuel injection pressure up to the highest wind tunnel ~~test~~ speed ($M = 0.85$). However the burning was not very smooth. The fuel consumption seems 7 lb/hr/lb of net thrust at $M = 0.8$.

2. Diffusers were tested at the supersonic windtunnel at LFA. It was found that with the usual duct opening a normal shock wave always forms ahead of the duct opening and the diffuser efficiency was rather poor. To improve the design, the entrance was made to be annular by introducing a central cone protruding ahead of the duct. The cone generated an oblique shock wave. The diffuser efficiency was then improved as the loss through an oblique shock is always smaller than the loss through a normal shock.

6.5.5
Aeropulse
脉动式空气喷气发动机

作者在1945年5月20日所写的题为“Aeropulse”（脉动式空气喷气发动机）的调研报告手稿共有7页。手稿叙述了德国人在研制脉动式空气喷气发动机方面的历史情况，并对他们在工程研制方面的主要经验作了一个简要总结。这里仅选印手稿的首页。

1935年，P. Schmidt在德国空军部的领导下，开始研制脉动式空气喷气发动机。第一步是设计和试验每秒50次的点火装置。空气阀设置在发动机的前方，这种空气阀的结构形式直到1945年均未作过什么大的变动；然而当初注入燃料的系统非常复杂，效果也不好。1939或1940年，柏林的Argus Motor Company（Argus发动机公司）开始研究脉动式空气喷气发动机，最初他们采用自己设计的空气阀，结构庞大而效果也不理想，但是他们的燃料注入系统十分简单。接着，他们吸收了Schmidt设计中的优点，放弃了Schmidt的复杂的注入系统和Argus发动机公司自己的庞大的空气阀，形成了作者写作那年所见到的脉动式空气喷气发动机。

约在1941年，德国空军部的Schelp看到这种发动机的潜力，建议将其用来推进小的无人轰炸机，那时V-2火箭还没有做出来。详细的空气动力学的性能研究是在Hermann Göring航空研究实验室的2.8米的高速风洞中进行的。

手稿中对德国人的主要研究经验谈到了以下两点。1. 关于空气动力学特性：一开始研制时，发动机是不带外罩的，风洞实验说明外部阻力很大。后来在空气阀上加上外罩，阻力明显降低。为了进一步增大推力，他们认识到必须扩大空气阀的有效进口截面积，准备做进一步的系统实验。2. 关于空气增压器：德国人设想如果在把燃料——空气混合物引入发动机之后，能够再单独地把空气引入发动机，那么当混合物发生爆炸(explosion)以后，会形

成一个气体活塞而把空气柱推出去。每次爆炸所推动的空气质量便增加了，从而增加了动量，以致提高了效率。P. Schmidt 的工作是以这一原理作为基础的，但据 A. Busemann 说，这方面尚未取得明显的效果。

2

Aeropulse

(H. S. Tsien)

May 30, 1945

I Historical Development P. Schmidt started to work on the aeropulse in 1935 under the auspices of the German Air Ministry. His first step was the design and testing of an ignition device to give 50 cycles/sec. This device is mechanical and the ignition in itself was achieved by the compression of a fuel-air mixture with a free piston. The air valve in front of the tube was essentially of the form used now. However the injection device for fuel was very complicated. It consisted of a very large number of wire gauge covered orifices with fuel pressure applied against the flow direction. It was claimed that by so doing, a constant fuel-air ratio could be maintained for all air velocities. During the initial run of this design, it was found that although the ignition device was giving 50 ignitions per second, the tube itself was operating at 100 cycles per second which is the natural frequency of the tube. The ignition device is thus unnecessary and was discarded.

In 1939 or 1940, the Argus Motor Company independently started their work on the aeropulse, first with an air valve of their own design. This design consisted of a special air passage which has much less resistance for air to go into the motor chamber than for air to go out of the motor chamber. The fuel injection system was, however, very simple and consisted of a single orifice. The frequency was approximately 50 cycles per second. Later on the good features of the Schmidt and the Argus designs were combined: the complicated injection system of Schmidt and the bulky Argus air valve were discarded, and the aeropulse took practically the same form as it is now.

About 1941, Schelp of the German Air Ministry saw the potentiality of the engine and suggested the application to propel a small unmanned bomber, since the development of V-2 was not ready. The air frame

6.5.6
Liquid Explosive Bombs
液体炸药炸弹

作者在1945年5月20日所写的“Liquid Explosive Bombs”（液体炸药炸弹）的调研报告手稿只有1页。手稿叙述了战时德国人在研制液体炸药炸弹方面的试验情况。这里选印了这份手稿。

德国人在1942年开始研究用液体炸药制造炸弹，所用的汽油加四氧化二氮混合而成的液体炸药所释放的能量，比等重量的一般固体炸药高出50%。为了保证安全，作为燃料的汽油和作为氧化剂的四氧化二氮分盛在两个容器内，容器之间被一个固定的空间分隔开来以保安全。当炸弹在飞机上投下以后，由压缩空气将汽油加压，使其通过许多喷管而喷入液态的四氧化二氮。经过10秒钟后达到完全的混合，然后在炸药接触地面时，引发普通引信而发生爆炸。

德国人的一个重要发现是，不要在四氧化二氮中混有硝酸，因为硝酸和汽油混合时，在有四氧化二氮参加的情况下，会产生足够的热量，不需要引信而自行爆炸。这一事实被用来解释试验中出现的事故。

他们做过许多次飞机投弹试验，后来又做了爆炸性能的精确测量。

1

Liquid Explosive Bomb

(H.S. Tsien)

May 20, 1945

I. Introduction

It was calculated that by using gasoline and nitrogen tetroxide N_2O_4 mixture, the heat value can be increased by 50% over the usual high explosive on equal weight basis and by 20% on the equal volume basis. However such a mixture is not safe to handle, so the mixing is actually accomplished after the fuse is armed. This is possible with liquids.

The work on this type of bomb was started at the test station at Fassbung (near Munden) in 1942. But there were many interruptions due to accidents with loss of personnel. Similar work was carried on at Heeresanstalt Peenemünde with different liquid combinations.

II. Description of Test Bomb

The oxidizer component N_2O_4 of the bomb has a very narrow range of temperature for which the material exists as a liquid at ordinary pressure. By some preconization, this temperature range is extended. The whole range is then shifted to lower temperature by addition of a few percentage of a second material. The fuel component is gasoline. The ratio of the two components is on the rich mixture side, i.e., contains more fuel than the stoichiometric mixture.

The two compartments containing these liquids are separated by a dead space so that the chance of piercing both compartments by a bullet is reduced. After the bomb is released from the aircraft, the compressed air in a small container pushes the gasoline through a series of nozzles to spray into the liquid N_2O_4. The mixing is estimated to be completed in 10 seconds. Then the bomb is exploded by the usual fuse upon contact with ground.

6.5.7

Installation of Turbojets in an Airplane

飞机上涡轮喷气发动机的安装

作者在 1945 年 5 月 21 日所写的题为 " Installation of Turbojets in an Airplane " (飞机上涡轮喷气发动机的安装) 的调研报告手稿共有 4 页。手稿介绍战时德国针对涡轮喷气发动机应该安装在飞机上的什么位置这一问题所做的研究工作。这里仅选印手稿的首页。

在飞机上安装一般的发动机和推进系统，总是想找一个最优位置，尽量减小因为他们和机身互相干涉而造成阻力的增加，也尽量避免对起控制作用的翼面产生不良影响。由于涡轮喷气发动机喷出的射流具有高的速度和温度，对飞机的其他部件的影响相当独特，于是安装问题成为涡轮喷气发动机设计中的一个最为重要的问题。

研究安装问题，风洞试验是最为方便的方法。德国人的大多数试验是在哥廷根空气动力学实验室 [Aerodynamische Versuchstalt Göttingen (AVA)] 进行的。他们选择的模拟方案是：采用电动机带动风扇，压缩进入发动机模型的空气，再燃烧酒精对压缩气流加热，最后排出模型。因为只要求中等的排气速度，只需用单级风扇，便可得到平稳的燃烧。

首先要问燃烧加热是否必要？如果通过涡轮喷气发动机的空气动力学特性是关键所在，那么加热不是必要的。但是，由于冷的射流和热的射流的扩展过程不同，包括射流的尾涡（诸如尾部的表面特征）在内的空气动力学特性的研究，只有在热射流试验中才能精确地进行。

空气动力学特性试验的内容包括发动机本身的特性，如升力、力矩和推力等，以及发动机和机翼的干涉阻力。在试验中，A. Busemann 指出了一个有趣的事实。他说，发动机喷出的射流把周围大约 8 倍射流直径范围内的空气连续地卷混在一起，所造成的具有一定频率的涡旋给尾翼的颤振带来麻烦。

德国人也对发动机进口的设计方案下了功夫，经过试验研究，总压头的损失减小到 10% 。

INSTALLATION OF TURBJETS IN AN AIRPLANE

(H. S. Tsien)
May 21, 1945

I. Introduction.

The installation problem of the conventional engine and propeller propulsive system consists of finding the optimum location of the power plant so that the increase in drag due to interference will be a minimum and no undesirable influence will be exerted on the control surfaces. Of course the same problems also exist in the case of turbjet powered airplanes. Due to the high velocity of the jet and the high temperature of the exhaust gas, the influence on other parts of the airplane is even stronger than in the case of the conventional power plane. Therefore the installation problem is one of the most important problems in jet airplane design.

II. Simulated Models of Turbjets for Wind Tunnel Testing.

To study such installation problems, wind tunnel testing is the most convenient method. Most of the study done in Germany was made by the staff at the Aerodynamische Versuchsanstalt Göttingen (AVA). The first thing to be determined is, of course, the best way of simulating the turbjets in model tests. The AVA (Ref.1) simulated the turbjets by a combination of electric motor driven fan and heat addition by burning alcohol. The air entering the model duct was compressed by an axial fan, then the compressed air was heated by burning with alcohol and discharged out of the duct. The fan in the duct was driven by an electric motor. Since only moderate discharge velocity was required due to low wind tunnel velocity compared with flight velocity, the fan was of single stage. Alcohol was chosen as the fuel for smoth combustion.

The first question to be settled was whether the heat addition by burning is absolutely necessary. Of course the answer is conditioned by the particular aerodynamic characteristics to be studied. If the flow characteristic around the turbjet is the essential point, then it was found that heat addition is not necessary. Accurate enough results can be obtained if one makes the momentum charges from inlet to outlet of the duct equal for both cold jet and hot jet. The later Göttingen tests were generally made with cold jets. However, due to the difference in the spreading of cold jets and of hot jets, studies on the aerodynamic characteristics involving the wake of the jet (such as tail surface characteristics) can only be accurately made with a hot jet.

If the momentum increases of the cold jet and the hot jet are made to be equal, the mass flows will not be the same. This situation can be remedies by

(a) Proptionally decreasing the exit area of the cold model so that both momentum charge and mass flow will be the same.
(b) Introducing a gas of lighter density into the duct to reduce the density of the exhaust from the cold model.

6.6
为著名报告《Toward New Horizon》（迈向新高度）所写的材料

1945年以Theodore von Kármán（冯·卡门）为首的科学咨询团为美国陆军航空兵（The Army Air Force）完成了题为《Toward New Horizon》（迈向新高度）共9卷的带有展望和规划性的报告，为二次大战结束以后美国空军的现代化建设提供了远景发展蓝图。钱学森为《Toward New Horizon》提供了他自己的观点和思想。他在1945年5月所写的对德考察的调研报告的基础上，总结了欧洲国家的研究经验，并且结合战时美国的研究情况，特别是他和他的同事们在加州理工学院和喷气推进实验室所做的工作，在《Toward New Horizon》这一研究报告的第3、4、6、7和8卷以及技术情报附录中，钱学森详细地论述了有关高速空气动力学、脉冲式空气喷气发动机、冲压发动机、火箭、超声速箭形翼导弹以及核能作为飞行动力的可能性等方面的研究概貌、存在问题以及发展前景。

保留的手稿中有关于高速空气动力学、脉冲喷气发动机和火箭等三方面的打字稿，下面分别选印其中的一部分。

6. 6. 1

High Speed Aerodynamics

高速空气动力学

这份题为" High Speed Aerodynamics "（高速空气动力学）的报告的打字稿共有28页，报告分成六节，每节的标题分别是：Ⅰ. 空气动力学中空气可压缩性的影响；Ⅱ. 维持层流边界层以减小阻力；Ⅲ. 击波以及击波和边界层的相互作用；Ⅳ. 临界飞行马赫数的控制；Ⅴ. 从推进动力装置喷出的射流对飞机周围流动的影响；Ⅵ. 跨声速和超声速飞机的设计问题。这里仅选印打字稿的首页。

为了实现跨声速和超声速飞行，必须克服因飞行速度增大而引出的困难，其中之一便是所谓"声障"，这份报告集中讨论与"声障"有关的空气动力学问题，指出了今后研究的方向。

当飞机的飞行速度不断提高到接近空气中的声速时，发生阻力骤增升力骤减以及压力中心后移，造成飞行稳定性恶化的现象。为了减小阻力，针对阻力的两个来源，即摩擦阻力和压差阻力，作者分别建议研究以下两个方案：(1) 建议研究层流翼型。可以在翼面上开槽以便将边界层中一部分气流吸入槽内从而减小阻力；(2) 建议研究后掠翼型，可以提高临界飞行马赫数的数值，力求避免在流场中出现击波。

报告还讨论了击波和边界层的相互作用，从喷气推进装置喷出的高速高温射流对于绕过飞行体的主流所产生的影响，以及控制面（如阻流板 (spoilers)、铰折面 (hinged surface) 等）的设计等问题。

解决上述问题不仅理论计算有困难，风洞试验也有困难，报告中作者分析了风洞的试验段的壁面对飞行体模型的干扰作用、机翼与机身的相互干扰等难题，实际上这些问题即使在今天看来也是我们所熟知的棘手课题。

HIGH SPEED AERODYNAMICS

(Draft of Section A of Part III of the Final Report)

Hsue-shen Tsien

I. The Effect of Compressibility of Air in Aerodynamics

When a body moves through the atmosphere, the effect of its motion on the surrounding air can be considered as that caused by a disturbance. Since any disturbance transmits with the velocity of sound which itself is nothing but a series of small disturbances, the signal for the motion of the body is also propagated throughout the medium with the velocity of sound. If the body moves very slowly, then in the time scale of the motion of the body, the signal velocity is practically infinitely large. In other words, the disturbance is felt almost "instantly" (referred to the time scale of the motion of the body). This means that the fluid medium, the air, can be considered as incompressible and hence no appreciable, elastic adjustment is present to take up the time of propagation. Therefore for slow motion the air can be considered as incompressible and this forms the basis of all classical aerodynamics.

As the speed of motion of the body is increased, the time of propagation necessary for the disturbances or signals can no longer be neglected, i.e., the elasticity or the compressibility of the air must be taken into account. Here it is immediately clear that the measure of the effect of compressibility is the ratio of the speed of body and the velocity of sound in the fluid, i.e., the Mach number. In other words, if the Mach number is small, the air can be considered

6.6.2

Aeropulse

脉冲式空气喷气发动机

这份题为"Aeropulse"(脉冲式空气喷气发动机)的报告的打字稿共有14页，报告分成四节，每节的标题分别是：I. 脉冲式空气喷气发动机的现状；II. 对现有脉冲式空气喷气发动机形式的可能改进；III. 无阀型脉冲式空气喷气发动机；IV. 结语。这里仅选印打字稿的首页。

这份报告在说明了脉冲式空气喷气发动机的简单原理之后，介绍了德国人在第二次世界大战期间，把这类发动机首先应用在飞弹上的主要性能，包括为提供单位推力在单位时间内所消耗的燃料量（简称比耗，以 lb/hr/lb 为单位）、脉冲频率和推力等参数值、以及这些参数随飞行马赫数的变化情况。但是德国人没有提供马赫数超过 0.6 的高速飞行的性能，于是作者对高速情况下的发动机性能作了简化分析和计算，估算出德国发动机的比耗不会低于 3 lb/hr/lb；用很简单的风琴管的振动模型近似估算了脉冲频率；在德国人给出的低速实验数据的基础上进行外推，导出了在超声速情况下推力系数几乎不随飞行马赫数变化的结论。

作者指出，可以在增加推力、降低燃料比耗方面进行改进，为此提出了一些具体的改进方案，如加大空气流进燃烧室的有效截面，在空气——燃料混合物爆炸以后再将空气单独引入发动机，把发动机装入机身内部等。这就要求研究和开发性能更好的材料和结构形式，使发动机更轻和更有效。作者提出了一个称之为"Valveless Aeropulse"的崭新的无阀型脉冲式空气喷气发动机的方案。他认为在高速飞行时，高速空气流所具有的惯性足以起到阀门的作用而没有必要采用机械阀，经过分析，相信无阀型发动机的推力系数可以比德国人的带有弹簧阀和文托里管的有阀型发动机的推力系数提高40%。

为了实现上述改进，需要展开进一步的研究，特别是实验研究，作者提出必须要有一个试验段足够大的超声速风洞，在里面可以做带有燃烧的完整的发动机模型试验，考察发动机的外部和内部的强烈脉冲流动的规律，进行静态试验或只是空气流经发动机的试验是不可靠的。

SECRET

AEROPULSE

(Draft of Art. 7, Section C, Part III)

Hsue-shen Tsien

I. The Present Status of Aeropulse

The German aeropulse and the American copy of it for the flying bombs is the first successful realization of this type of power plant. The general dimensions are given in Fig. 1. The air is sucked into the combustion chamber by the vacuum created by exhaust of the previous cycle. The intake air passes the venturi where gasoline is continuously injected. The explosion of the air fuel mixture raises the pressure in the combustion chamber to a high level and closes the spring valve at the intake. The gas is thus forced to expand through the exhaust duct and discharged at high speed. This gives the propelling impulse. At the end of discharge, the inertia of the gas will create a vacuum in the combustion chamber and the engine is ready to start a new cycle again. The pressure in the combustion chamber is controlled by the rate of fuel injected and this in turn controls the discharge velocity of the gas and thus the propulsive thrust. To start the engine, a carefully adjusted amount of fuel is sprayed into the cold combustion chamber so as to create a mixture of correct ratio around the spark plug. The spark plug ignites the mixture and the resultant strong explosion starts the cycle. The flow in the combustion chamber and the discharge duct is thus a pulsating one with very large amplitude as shown by Figs. 2 and 3.

SECRET

6.6.3

Rockets

火箭

这份题为"Rockets"（火箭）的报告的打字稿共有28页，报告分成四节，每节的标题分别是：I. 火箭的类型和目前的应用；II. 固体推进剂火箭；III. 液体推进剂火箭；IV. 结语。这里仅选印打字稿的首页。

报告一开始说明了固体推进剂火箭和液体推进剂火箭两类火箭的工作特性和用途。一般说来，固体火箭适合应用于工作持续时间短的情况，而液体火箭则适用于长时间的情况。

接着，报告分别讨论了固体火箭和液体火箭的发展现状，并提出了可能改进的方案。

在固体推进剂火箭部分，首先讨论了推进剂的密度、温度敏感性、燃烧表面积以及燃烧速率定律中的指数n等对火箭性能的影响。作者分析指出：密度较大的推进剂会使发动机较轻；若温度敏感性较高，发动机的工作温度范围将受限制；发动机中推进剂装得很满的受限燃烧可给出高比冲；小的指数n会有好的重复性和可靠性等。然后，作者论述了改进设计以求达到减轻发动机重量和提高排气速度方面的重要性，指出这和改进推进剂性能（如热值）所得到的好处同等重要。作者详细讨论了为减轻重量可能采取的措施，包括采用：燃烧压力低而燃烧平稳的、燃烧速率指数小、温度敏感性低的推进剂，燃烧速率高的推进剂进行受限燃烧以及其他措施。作者认为，研究和实现这些措施将会大大增加固体推进剂火箭的实用性。

在液体推进剂火箭部分，作者讨论了三方面的问题，即火箭的设计原则和推进剂的选择、推进剂的供给系统以及发动机的构造和设计。液体火箭主要应用于大型飞机的加速起飞和大型导弹的推进。前者的设计准则要求火箭工作可靠、使用简便以及有重复使用的长寿命；而后者则要求比冲

高、设计简单、操作容易以及生产成本低。作者给出了几种推进剂的性能。分析指出：要研制能瞬时点火并适用于温度范围较广的推进剂，要注意推进剂密度与飞行体体积和所受阻力的关系以及采用推进剂作为燃烧室的冷却剂所要求的条件等。作者分析比较了各种推进剂供给系统，认为压缩气体系统适用于工作时间短而推力大的火箭，涡轮泵系统适用于工作时间长的火箭，至于将来要大量使用的导弹武器，气体发生器系统则更为可取，因为它结构简单、优点很多，但要考虑生产成本。作者建议今后要大力研究后面两种供给系统。作者又论述了四种驱动泵的动力系统的方案（即：与辅助发动机或与飞机主要动力装置相连；使用由推进剂驱动的另一个气体涡轮；使用旋转式火箭发动机；以及风车等）的优缺点，指出了各自适合的用途以及需要进一步研究的问题。关于发动机的构造和设计，作者提出了两个主要问题，一是燃烧室的形状应能使注人的推进剂进行最有效的燃烧，为此今后需要深入研究燃烧室内的流动图案；另一是对长时间工作的发动机的冷却要做到既有效又经济。他提到了两种冷却方案，一是用推进剂流过燃烧室和喷管的外套实现对流冷却，另一是将推进剂直接注人燃烧室内壁形成液膜，或通过多孔材料作成的燃烧室壁而注人燃烧室实现蒸发冷却，对此今后需要进行深入的比较研究。

最后，作者作了扼要的总结，认为：固体推进剂火箭今后的发展更多地依赖于推进剂性能的改善；而液体推进剂火箭则更多地依赖于机械设计的研究。今后为了设计用于重型导弹的高效大型火箭，特别需要在燃烧、冷却、输送用泵等问题上开展强有力的工作。

这份报告自始至终体现了作者对待一个大型工程研究项目的理论与实际紧密结合的风格，既注意深入的理论分析，指出关键性的研究课题，又重视与设计和制造以及生产经济性等有关的综合性很强的实际问题，并提出可能改进的措施。

ROCKETS

(Draft of Art. 9 of Section C of Part III)

I. Types of Rockets and Their Present Applications

The two main types of rockets are the solid propellant type and the liquid propellant type. The solid propellant rockets are now used or suggested to be used for propelling the artillery rockets, for the assisted take-off of aircraft, for the launching of flying bombs and missiles, and for the propulsion of large missiles. The liquid propellant rockets are used for the assisted take-off and for the propulsion of very large missiles and airplanes. While there is no essential difference in the operating characteristics of these two types of rockets and thus for any new application the possibility of both types should be investigated, there are certain facts which should be kept in mind. The solid propellant rocket contains all the propellant in the high pressure combustion chamber or the motor. Thus if the duration of the operation is long, the chamber volume becomes very large and the weight of the chamber will be very large. Therefore, for very long durations, i.e., durations in excess of 30 or 40 seconds, the weight of a solid propellant rocket is heavier than that of a liquid propellant rocket. However, this line of demarkation also depends upon the thrust of the rocket. The reason for this variation is that for the liquid propellant rocket the unit weight is a function of the thrust. Larger thrust makes the unit weight smaller, especially in the case of pump fed rockets. The above value of 30 to 40 seconds corresponds to a thrust of approximately 4000 lbs. In other words, for durations in excess of 30

6.7
课程讲义和大纲

6.7.1
High Temperature Design
高温设计

1949－1950年，钱学森与P. E. Duwez合作为美国加州理工学院喷气推进中心的研究生开设了"High Temperature Design"（高温设计）一课，课程编号为JP 210。课程内容分为热应力和高温材料两大部分，由作者负责第一部分，而Duwez负责第二部分。这里选印讲义手稿中第1和第21两页，以及该门课程的期终考题。

第二次世界大战结束后，喷气推进技术发展很快，已经进入实用阶段。发动机和飞行体的很多部件都在高温下工作，进行设计时应当充分利用材料在高温下的极限性能，所以研究和说明材料所经受的工作条件就变得很重要，其中的一个环节就是要决定材料在随时间变化的加热情况下所承受的热应力的变化，在这个基础上选用合适的材料并制定合理的设计方案。讲授这门课程的目的就是要给出求解这类热应力问题的方法。

本课程的第一部分的内容便是介绍求解热应力的方法。求解分为两步：第一步在给定外界对物体的加热条件下，求解一个物体内部的热传导问题，计算出物体在每一时刻的温度分布；第二步在已知温度分布的基础上，考虑到物体的热涨冷缩的效应与变形和受力之间的关系，计算物体在每一时刻的热应力分布。课程的第二部分则提供材料在高温下的性质，包括弹性、塑性、蠕变、拉延性、疲劳、冲击热阻等，也介绍了一种先进的高温材料——钛的高温特性。

2

40 copies

JP 210

High Temperature Design Problem

Part I

Thermal Stress

Introduction

As the limits of maximum performance of material at high temperature are approached in jet propulsion designs, it becomes important to understand and to be able to specify the working condition the materials are subjected to. One very important phase of this study is the determination of thermal stress in the material under non-steady heating. For instance, for ceramic materials, we usually speak of the "thermal shock resistivity", a property which indicates the merit or the degree of sudden heating the material can stand without breakage. Clearly this is a thermal stress problem. We shall not be able to specify the desired material till the the thermal shock resistivity is expressed in terms of more basic material properties such as Young's modulus, Poisson's ratio, heat conductivity, specific heat and the coefficient of thermal expansion. To give methods for solution of such problem is the purpose of the present series of lectures.

The analysis of thermal stress is divided into two steps:

1) The computation of temperature distribution in the material at different time instant with the specified boundary conditions and initial conditions. Boundary conditions are the condition of heat flux at the boundaries of the material. Initial conditions are

It is interesting to note that the temperature at the center of the disc is not zero (or the temperature at center is not the cooling air temperature) but higher.

Exercise 2   Take $q = 0.6$ B.t.u./in², sec.
$b = 1.6$ inches
$R = 10.5$ inches
and values for h and k from Exercise 1, compute the temperature distribution in the disc.

p.21 Thermal Stress

The theory of elasticity of engineering materials is based upon the following assumptions:

a) The material is isotropic so that the relation between stress and strain is independent of the coordinate system or invariant under coordinate transformation.

b) The stress strain relation is linear and no permanent strain exists

The first assumption is based upon the fact that although the individual crystals of the material are not isotropic, the aggregate of the small crystals in random orientation is isotropic if the volumes considered contains many (MANY) crystals. The second assumption is based upon small stress and small strain so that no plastic deformation occurs.

Besides these two assumptions, we need, of course, the Newtonian condition of force equilibrium and the hypothesis of continuum (CONTINUUM) in that no material is created nor destroyed.

June 1, 1950

FINAL EXAMINATION ON JP 210

(1) A uniform thin rod of length ℓ and cross-section A is heated to a non-uniform temperature (non-uniform along the rod, but can be considered as uniform in each cross-section) and then after the heating has stopped the temperature at a number of stations along the rod are measured at several time intervals. Consider the rod to be thermally insulated after the heating and consider the heat capacity of the material of the rod to be known, suggest a practical method of analyzing the temperature measurements to obtain the heat conduction coefficient k.

(2) Consider a thin-walled long cylinder of radius R and thickness b, closed at ends and subjected to internal pressure p and a heat flux density q from inside to outside. Steady temperature distribution in the wall is maintained by cooling the outside surface to room temperature. Assume that the material behaves as an elastic body with Young's modulus E and Poisson's ratio ν . The tensile strength of the material is σ^*, and the coefficient of thermal conductivity k, the linear thermal expansion coefficient α .

Compute the stress distribution in the cylinder, and the maximum stress. Show that with other variables held constant, there is an optimum value for the thickness b such that the maximum stress is the lowest. Then show that for a given material, there is a maximum value of the product qRp beyond which no design is possible. Determine this critical (qRp) in terms of the material properties listed above. Discuss the comparative merit of the different classes of material for this application.

6.7.2
The P－L－K Method
PLK 方法

1954 年，作者在美国加州理工学院开设了“PLK 方法”的系列讲座，撰写了题为“The P(oincare)—L(ighthill)—K(uo) Method”的讲义。讲义手稿共有 82 页，内容分为引言、常微分方程、双曲型偏微分方程以及椭圆型偏微分方程等四章，最后一页是志谢页。这里选印讲义手稿中引言一章中的前 3 页及志谢页。

在本世纪的 40－50 年代，人们为实现高速飞行而致力于突破“声障”的研究。1949 年，作者离开美国麻省理工学院，重返加州理工学院任教。在他驱车西行途中，到康乃尔大学与挚友郭永怀相聚，得知郭永怀已对跨声速气体动力学提出了一个新课题，即击波与边界层的相互作用问题。这一问题极难，不仅在于微分方程的非线性，而且因为边界层的前缘有奇点，采用线性化的近似方法或者一般的摄动法都不能解决问题。郭永怀独辟蹊径，把 Prandtl（普朗特）的边界层理论和 Lighthill（莱特希尔）的变形坐标法结合起来，形成了一种独特的奇异摄动法，消除了边界层前缘的奇异性，得到了一致有效的流场解，在这一问题上取得了重大成果。作者认为郭永怀之所以取得成功是他治学严谨，有见识，有胆量，敢于和善于攻坚的结果。到了 1953 年冬，作者再次见到了来加州理工学院讲学的郭永怀，有机会向他的老友学习奇异摄动法。此后，作者对这一方法进行了系统的整理和研究，在 1954 年的三四月间写出了本手稿，把这一方法命名为“P(oincare)—L(ighthill)—K(uo) 方法”，其中第三个字母 K(uo) 便是老友的姓——郭，我们可以透过这一命名体会到作者和他的挚友之间的亲密的关系。手稿不仅深入浅出地、出色地阐述了这一方法，宣传了这一方法的实质和应用，而且将这一方法推广到求解某些具有奇点的椭圆型偏微分方程的问题。同年，作者为研究生讲授了“PLK 方法”一课；次年，即 1955 年作者在《Advances in Applied Mechanics》（应用力学进展）的第

4卷上发表了同名文章，对奇异摄动法的推广应用与后来的发展起到了重要的推动和促进作用。时至今日，奇异摄动法仍然是我们求解非线性方程的一个现实而有力的解析手段。

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1-1

The PLK Method

Chapter I

Introduction

1.1 Historical Development

In the famous work, "Les méthodes nouvelles de la mécanique céleste" (1892) (Chapter III), H. Poincaré devised a method for finding the periodic solution of a system of first order equations,

$$\frac{dx_i}{dt} = X_i(x_1, x_2, \cdots x_i, \cdots x_n; \alpha) \quad (i = 1, 2, \cdots n) \qquad (1.1)$$

where t is the time variable and α is a small parameter. The equations with $\alpha = 0$ are particularly simple and a periodic solution with period $T^{(0)}$ can be easily found. The essence of the method is the expansion of the solution in the parameter α, not only the variables x_i

$$x_i = x_i^{(0)} + \alpha x_i^{(1)} + \alpha^2 x_i^{(2)} + \cdots \qquad (1.2)$$

but also the period T,

$$T = T^{(0)} + \alpha T^{(1)} + \alpha^2 T^{(2)} + \cdots \qquad (1.3)$$

In recent years, this method has found many applications in the theory of non-linear oscillations for there the same equations as Eq. (1.1) prevails. However for nearly sixty years no extension of the principle of this method was made, and the full potentiality of Poincaré's invention unrecognized.

On May 19th, 1949, M. J. Lighthill gave a lecture before the London Mathematical Society on "A technique for rendering

approximate solutions to physical problems uniformly valid" which introduced a very important extension of Poincaré's method. Lighthill's objective is to improve the well-known method of perturbation for calculating the approximate solution of a physical problem. Such perturbation method is based upon the concept of expanding the exact solution in a power series in the small parameter α, the zeroth order solution being independent of α, the first order solution proportional to α, etc. In many practical problems, however, the zeroth order solution may contain a singularity at a point or a line within the domain of interest. Then not only the singularity will appear at the same location again in the higher order solutions, but worse, the singularity will become progressively more severe as the order of the solution increases. Thus the power series expansion in α breaks down near such singularities. Lighthill's method is designed to eliminate such difficulties and to render the expansion uniformly valid over the whole domain. All this is accomplished by expanding the independent variables or coordinates also in power series of the parameter α. For instance, if the coordinates are x, y and the variable is u, then

$$u = u^{(0)}(\xi,\eta) + \alpha u^{(1)}(\xi,\eta) + \alpha^2 u^{(2)}(\xi,\eta) + \cdots \qquad (1.3)$$

$$\left.\begin{aligned} x &= \xi + \alpha x^{(1)}(\xi,\eta) + \alpha^2 x^{(2)}(\xi,\eta) + \cdots \\ y &= \eta + \alpha y^{(1)}(\xi,\eta) + \alpha^2 y^{(2)}(\xi,\eta) + \cdots \end{aligned}\right\} \qquad (1.4)$$

It is found that $u^{(0)}(\xi,\eta)$ is simply the zeroth order solution of the classical perturbation method with ξ, η replacing x, y. If we neglect the higher order terms in u of Eq. (1.3), then the

approximate solution is simply the zeroth order perturbation solution with the coordinates stretched or distorted by the transformation equations (1.4). For many practical applications, this is usually all that is required. The stretching of coordinates improves the zeroth order solution to such an extent that the result is nothing short of being a wonder.

Lighthill applied the method to problems involving partial differential equations where the zeroth order solution is obtained from a reduced linear equation of equal order as the exact equation. Y. H. Kuo (1953) demonstrated that the principle of solution can be even extended to problems where the zeroth order equation is an equation of lower order than the exact equation, as is the case of the boundary layer theory. Kuo also used in the method for complicated problems involving junction of regions of solution with different coordinate distortions.

It is thus appropriate to call this very powerful method in applied mathematics the PLK method.

1.2 Simple Example

To illustrate the principle of PLK method, let us consider the following first order ordinary differential equation:

$$(x+\alpha u)\frac{du}{dx} + u = 0 \qquad (1.5)$$

By dividing the equation with du/dx, we have

$$u\frac{dx}{du} + x = -\alpha u$$

Or

$$\frac{d}{du}(xu) = -\alpha u \qquad (1.6)$$

Acknowledgement

The author of these Notes wishes to thank his colleagues for encouragement and helpful discussion during the course of a series of seminars on the title subject during the spring of 1954. In particular to Professor A. Erdélyi, the author is deeply indebted for his many kind critical comments on mathematical matters, and for his generosity in contributing the Appendix to Chapter II. To Professor Y. H. Kuo, Graduate School of Aeronautical Engineering, Cornell University, the author wishes to express his gratitude for initial introduction to this subject and for many enlightening conversations on the problem of supersonic boundary layer flow.

H. S. Tsien

April, 1954

6.7.3

“Aerodynamics（空气动力学）”、“Rockets（火箭）”、“Aeronautics Seminar（航空学专题讨论）”等课程的课程提纲或说明大纲

在作者的手稿中，有一部分是作者为学生们讲授“空气动力学”等课程所写的讲稿。为了节省本选集的篇幅，在此仅选印两页课程提纲的打字稿以示梗概。一份是“空气动力学”一课的提纲，内容涉及到跨声速、超声速和高超声速的可压缩性流体动力学以及稀薄气体动力学；另一份是作者预见到今后航空和火箭技术飞速发展的需要，建议为航空工程系的学生开设“火箭”课和“航空学专题讨论”课的说明大纲。上述文稿的完成时间不详。

Course Descriptions

16.051 Aerodynamics - Compressible Fluids, I (A). Characteristic parameters for flow of compressible fluids. One-dimensional flow—Delaval nozzle. Shock wave; change of states across a shock wave, formation of shock wave and its thickness. Basic equations for two and three dimensional flow of non-viscous compressible fluid, Kelvin's theorem, Helmholtz theorem. Two-dimensional subsonic flows: Jansen-Rayleigh method, Glauert-Prandtl method, Karman-Tsien method.—Velocity and pressure correction formulae. Exact solutions of two-dimensional flow, limiting line, critical Mach number. Transonic flows and Karman transonic similarity law. Two-dimensional supersonic flows: the method of characteristics, Ackeret's theory for thin supersonic airfoils.

16.052 Aerodynamics - Compressible Fluids, II (A). Linearized theory for three-dimensional subsonic flows; axially symmetric bodies, velocity and pressure correction formulae; induced velocity of a vortex and extended Prandtl lifting line theory for wings of large aspect ratio. Transonic flows over axially symmetric bodies—Karman similarity law. Linearized theory for supersonic wings of finite span. Method of source and doublet distribution, drag and lift of symmetrical rectangular wings, drag and lift of symmetrical delta-wings. Hypersonic flows:—its characteristics and the hypersonic similarity law. Basic equations of viscous compressible flow: Phenomenological derivation and its justification by kinetic theory of non-uniform gases.—Superaerodynamics. Boundary layer in compressible flow and its interaction with shock wave.

Proposed Courses in Aeronautical Engineering

16.57 Rocket (A). Principles of operation of rockets. Combustion and thermodynamical equilibrium. Solid rocket propellants. Fundamentals of solid propellant rockets and its design. Liquid rocket propellants. Liquid propellant rocket and its design. Rocket trajectory, principles of its calculation. Long range rockets, multi-step rockets.

Prerequisite	16.02
Year	G(A), elective
Term	1st
Lecture Hours	3 per week
Preparation Hours	6 per week
Instructor	Tsien

16.61 Aeronautics Seminar (A). Weekly two hour seminar given by the staff and the graduate students to discuss current aeronautical problems. Recommended for all graduate students in aeronautical engineering.

Prerequisite	Graduate standing
Year	G(A)
Term	1st or 2nd
Lecture hours	2 per week
Preparation	2 per week
Instructor in Charge	Tsien

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